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Essays in public economics and mathematical finance

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Essays in public economics and mathematical finance

by

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DEDICATION

To

baba, maa,

maman,

and

of course,

Kavita.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

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ABSTRACT

0.1   Chapter 1 Abstract

Why do some economies remain technologically backward even when technologies on the frontier are available for adoption, virtually freely? If institutions are fragile and property rights insecure, potential adopters of frontier technologies may be dissuaded if adoption leads to increased expost conflict over rightful shares to the higher returns. In such a setting, publicly-funded protection of private property rights may successfully support the adoption of best-available technologies as a Nash equilibrium. The movement to more-secure property rights may or may not be welfare-enhancing.

0.2   Chapter 2 Abstract

In this chapter, valuation of a financial derivative, known as Stock Loan, is addressed when the underlying asset is subject to risk of bankruptcy. A stock loan is a financial derivative where the owner of an asset (a share of a stock) can obtain a loan from a lender (usually, a bank) using that asset as a collateral. The movements of the asset price is modeled to follow a geometric Brownian motion, with constant drift and volatility. Following the credit risk literature, risk of bankruptcy is introduced according to both *structural and reduced form approaches*. In the structural form modeling, default is introduced following the Black and Cox (1976) formulation, where the asset is declared as bankrupt as soon as the asset price falls below a pre-determined lower boundary. Modeling the lower boundary as a deterministic function of time, a closed form expression for the valuation of the financial derivative is obtained in terms of *the probability distribution of the first passage time of Brownian motion and the valuation of the Down-and-out barrier option*. The pricing formula in the structural form modeling is based on the celebrated Black and Scholes (1973) framework and therefore, is easy to implement. As a salient feature of structural form modeling, default time turns out to be a predictable stopping time. In the reduced form approach, bankruptcy is modeled to occur through a default intensity which is assumed to be a decreasing function of discounted stock price. The event of bankruptcy is modeled as a non-predictable phenomenon. In this formulation, the existence of an optimal exercise boundary is proved, which is of threshold type. This optimal decision threshold is crucially contingent on the policy variables that are treated as parameters of the system. We proceed further to use numerical methods to address the sensitivity analysis of the optimal exercise boundary. The results of our numerical simulation provide further insights into the linkage between optimal exercise boundary and the policy variables. We find that optimal exercise boundary is crucially contingent on the *effective rate of return* (defined as the difference between interest and lending rate) and exhibits a non-monotone relationship. We also find an interval where optimal exercise boundary shows a monotone increasing relationship with an increase in volatility. The sensitivity analysis in the reduced form modeling can be useful in recommending policy prescriptions in the valuation of mortgage backed securities.

CHAPTER 1.   Public Provision of Security in an Insecure Property Rights Environment

1.1   Introduction

The term property rights refers to an owner's legal right to use a good/asset for consumption or income generation and also, the right to transfer the good to another party. Property rights have received pride of place in all analyses of the development (and dominance) of the market system in modern societies. Over two centuries ago, Adam Smith and other thinkers expounded on the idea that property rights encourage their holders to develop the property, generate wealth, and efficiently allocate resources via the market mechanism.[1] They noted that the anticipation of profit from "improving one's stock of capital" rests on clear delineation and enforcement of private property rights, which, in turn leads to more wealth and improved standards of living for all.[2]

While the above prescription for material progress and prosperity has been around for over two hundred years, not every country has succeeded in using it to achieve sustained growth and development. Indeed, in most less-developed and transition economies, institutions aimed at defining and preserving property rights are woefully fragile, and as such, property rights are terribly insecure. This insecurity comes at a hefty price – heightened conflict over property and

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[1]A practical application of this principle can be found in the introduction of the Permanent Settlement System (around 1800) in colonial India. Under this system, the colonizers – the British under Lord Cornwallis, one of the leading British generals in the American War of Independence – granted proprietarial rights to former landholders (would-be zamindars) to the land they occupied. This method of incentivisation of zamindars was intended to encourage improvements of the land, such as drainage, irrigation and the construction of roads and bridges. The land tax was also fixed in perpetuity. Cornwallis successfully argued that "when the demand of government is fixed, an opportunity is afforded to the landholder of increasing his profits, by the improvement of his lands".

[2]Besley (1995) investigates the interconnection between investment and land rights using data from Ghana, when the country was in a state of transition between traditional and modern land rights. His findings for Wassa, a cocoa growing region where most of the land is owned, was supportive of the idea that "better land rights facilitate investment".

the accompanying dissipation of scarce resources in the creation of effective property rights.[3]

Our paper studies the consequences of insecure property rights on the mechanics of technological innovation. The work is motivated by a certain "social resistance" to technological change that characterizes many poor economies. For example, Platteau (2000, p.200) documents how fishermen in Congo refused to use a new net technology which was offered to them at no cost. More generally, it has been documented that economic agents in impoverished societies often reject superior technologies – technologies that are on the frontier – even when the cost of adoption appear negligible. In explaining this apparent paradox, Parente and Prescott (1999) make the convincing case that technological innovation is not a Pareto-superior outcome. There are economic winners and losers, and the latter have an incentive to block technology adoption by others because it necessarily influences the expost distribution of wealth. This view finds prominence in Olson (1982), Mokyr (1990), Krusell et al. (1996), among others.

Linked to this, is the view that post-production conflict is inevitable if the property rights are not perfectly enforced. Specifically, output is *contestable* in a society with imperfect property rights and conflict over the output cannot be settled without expending scarce resources in "appropriation" (grabbing the production of other agents or defending it from others). In the last two decades, a growing body of research has tried to explain the consequences of such conflict and appropriation in the process of development. Almost all of this work models conflict as a contest in which a non-cooperative game is played between agents to settle the conflict. A key ingredient of conflict is the use of weapons or defensive means, a composite form of which is termed "appropriative investment". Returns of appropriative investments that accrue to an agent is represented by "technologies of conflict"or "context success functions".

Continuing in this tradition, Gonzalez (2005) argues that the aforementioned paradoxical

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[3]In recent times, economists have popularized this line of thinking. De Soto (2000) has brought the argument into a broader public domain. Economic historians such as North (1981), Jones (1986), and Mokyr (2002) have cited evidence to support this view. There is a growing literature that focuses on the links between the security of property and economic behavior at the institutional level in a variety of specific institutional settings. For example Besley (1995), Goldstein and Udry (2008) study the impact of insecure land rights on investment and productivity in rural Ghana. In a related study Field (2007) finds that issuing of "property titles" in urban Peru has led to a significant increase in labor supply. Johnson et al. (2002) studies the impact of insecure property rights on the investment decisions taken by manufacturing firms in post-communist countries when bank loans were available. A common thread running through these studies is secure property rights facilitates the creation of wealth.

choice of inferior technologies can be understood as "a strategic response to the anticipation of conflict" over the expost distribution of newly-created wealth especially when property rights over it are insecure. Gonzalez (2005) has in mind a setting in which two agents contemplate adoption of a superior technology in an insecure property-rights environment. While each recognize that such adoption would lead to an increase in future output, each is nevertheless afraid that this newly-created wealth generates an incentive for the rival to engage in a costly game of predation. The expected predatory response discourages adoption of the superior technology in the first place, and thus "... poverty becomes the price of peace." (Bates 2001).[4] The upshot of the Gonzalez (2005) analysis is that adoption of the best-available technology is never sustainable as a Nash equilibrium.

If people are hesitant to adopt superior technologies because of a fear of subsequent conflict, would some sort of external intervention be beneficial? Would it help, if a third party intervenes in this conflict by providing some manner of public protection of rights on private property? To implement this, we introduce a "government" in the framework of Gonzalez (2005). We think of the government as imposing a non-distortionary tax on the initial endowments of each agent at the start of their life. The tax proceeds are utilized to finance the hiring of a "guard". The guard is simply a public security service whose sole aim is to reduce the effectiveness of each agents' predatory activities, without directly interfering in the expost conflict. The posting of a guard is shown to influence agents' decisions on allocation of resources to productive and predatory activities. In sharp contrast to the main result in Gonzalez (2005), we prove that adoption of the frontier technology by each agent can now be supported as a Nash equilibrium.

We go on to extend the analysis by allowing the government to directly influence the nature of the expost conflict. In other words, we allow the government to use its tax-financed resources to alter the existing regime of property rights. Presumably, a government can achieve increased security of property rights by funding the police, the judiciary, and the corrections systems better. We find that adoption of the best-available technology by each agent continues to emerge as a Nash equilibrium. Within this equilibrium, we find that improved property rights,

[4]Hall and Jones (1999) provide evidence that poor enforcement of property rights can be a serious impediment to technological progress.

though growth enhancing, is not always socially optimal from an aggregate-welfare point of view. [5]

The paper is organized in the following manner. Section 1.2 describes the benchmark model due to Gonzalez (2005). In section 1.3, we introduce the public security of private property and analyze the equilibrium outcomes. In section 1.4, we endogenize the property rights regime. Section 4 concludes the paper.

1.2   The model

1.2.1   Physical environment

We consider a two-period model of imperfect security of private property and its impact on technology choice. The model economy is inhabited by two agents, named R and P ("rich" and "poor") – these agents can be thought of either as individuals or collectives (such as tribes, nation states, and so on). There is a single good and the aggregate endowment of this good in period 1 is a fixed amount Y. Agent R is endowed with a share $p \in (1/2, 1]$ of Y; correspondingly, Agent P is endowed with the remaining share, $1 - p$. Rights to this property in period 1 are perfectly secure for each agent. However, property rights in period 2 are not secure, and all the action in this model derives from this insecurity.

Each agent uses a portion of his property in period 1 and undertakes some productive investment; the latter, via a production technology, produces consumables in period 2. At the start of period 1, each agent costlessly chooses a technology from a set of available technologies, $[A^L,\ A^H]$. A technology is to be interpreted as a blueprint that transforms investment into output in the following period. We assume that each agent has access to the same AK production technology and that productive investments of the agents are decided independently of each other. To be specific, productive investment K_i by agent i $[i \in \{R, P\}]$ at period 1 produces output $A_i K_i$ at period 2 where $A_i \in [A^L, A^H]$ is the technology choice of agent i.

In a world with secure property rights, the resources available to agent R in period 2 would

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[5]In a somewhat-related study, Gonzalez (2007) analyzes the growth-welfare trade-off in an exogenously-specified property rights environment. He showed a symmetric equilibrium allocation associated with more-secure property rights and faster growth can be Pareto dominated by one associated with poorer property rights and slower growth.

be $A_R K_R$, and that to agent P would be $A_P K_P$. Not so here. Here, the total amount of consumables ("common property") available at the start of period 2 is $Y' \equiv (A_R K_R + A_P K_P)$ and property rights over Y' is insecure, that is, it is subject to pillage and appropriation. This insecurity prompts agents to invest in appropriative investments that help convert their claims on production into effective property rights on the common output. Let X_i denote agent i's investment in appropriation, and let p' denote agent R's share of Y'; henceforth p' is labeled the "appropriation function". Then,

$$p \equiv \frac{(X_R)^m}{(X_R)^m + (X_P)^m} \in [0, 1]; \quad m > 0, \tag{1.1}$$

where (1.1) is a share function – taken as a primitive – capturing the technology of conflict over claims on future output. Note p' is increasing in an agent's own appropriative investment and decreasing in that of his rival's. This is the workhorse functional form for the technology of conflict. For future reference, note that p is symmetric and homogeneous of degree zero in X_R and X_P. This last property is analytically convenient and largely accounts for the widespread use of this functional form in the conflict literature. As an aside, note that resources allocated to productive investment in period one are not subject to appropriation, only the final output in period two is. Finally, note that if property rights were perfectly secure, agent R's share of Y' would be given by $A_R K_R / Y'$; therefore, as long as p in (1.1) deviates from this ratio, property rights are insecure. For future use, note that p in (1.1) can never approach $A_R K_R / Y'$. This last observation will make a major appearance in the penultimate section of this paper.

It is instructive to outline a time-line of events. At the start of period 1, each agent chooses a technology from the aforementioned set of available technologies. Once that is done, and cognizant of his own technology choice but not that of his rival's, an agent makes consumption, appropriation, and productive investment decisions, financing everything from his endowment. Production activity is then initiated. Agents consume and undertake the planned appropriation investments. When period 2 arrives, the common production, Y', is realized and agents receive their share which they consume; agent R gets a share p' and agent P, a share $1 - p'$. Note that p' is determined by *past* appropriation investments of both parties, as is described by (1.1).

The resource constraints in period 1 can be written as

$$pY = C_{1R} + X_R + K_R, \text{ for } i = R \tag{1.2}$$

$$(1-p)Y = C_{1P} + X_P + K_P, \text{ for } i = P \tag{1.3}$$

where C_{1i}, $i \in \{P, R\}$ is consumption by agent i in period 1. The second period constraints are

$$C_{2R} = p'(A_R K_R + A_P K_P), \text{ for } i = R \tag{1.4}$$

$$C_{2P} = (1 - p')(A_R K_R + A_P K_P), \text{ for } i = P. \tag{1.5}$$

where C_{2i}, $i \in \{P, R\}$ is consumption by agent i in period 2.

The description of the physical environment is complete once preferences are specified. We assume that agent i has preferences described by the separable utility function, $U_i \equiv \ln C_{1i} + \beta \ln C_{2i}$, $\beta > 0$.

1.2.2 Equilibrium

The aforediscussed time-line of events suggests the following characterization of the game. Period one is characterized by two stages, where in each stage, agents act non-cooperatively to maximize their payoffs without any information on their rivals' strategies. Therefore, we are faced with a two-stage game, where at each stage, agents play a simultaneous-move game, and the outcome of the first stage is not revealed before the actions of the second stage are taken. To find a reasonable solution, we look for the set of subgame-perfect equilibria. In other words, for any choice of technology at stage one, we first find the optimal consumption and investment strategies for each agent which are mutual best responses to each other. These optimal responses are solely a function of the technology choices made in stage one. Then, we incorporate these optimal decisions in the agents' utility maximization problem and find the set of technologies in stage one that produce non-cooperative optima for each agent.

Consider the problem faced by agent R at stage two of period 1. At this point in the game,

agent R knows A_R; he takes A_P, X_P and K_P as given, and solves the following problem:

$$\max \ U_R \equiv \ln C_{1R} + \beta \ln C_{2R}$$

subject to

$$pY = C_{1R} + K_R + X_R,$$

$$p'Y' = C_{2R},$$

$$p' = \frac{(X_R)^m}{(X_R)^m + (X_P)^m},$$

$$\text{and } Y' = A_R K_R + A_P K_P.$$

The interior optimality conditions for agent R are given by the following equations:

$$\frac{1}{C_{1R}} = \beta \frac{(X_R)^m}{(X_R)^m + (X_P)^m} A_R \frac{1}{C_{2R}}, \tag{1.6}$$

$$\frac{A_R}{A_R K_R + A_P K_P} = \frac{(X_P)^m}{(X_R)^m + (X_P)^m} \frac{m}{X_R}. \tag{1.7}$$

Equation (1.6) is a standard intertemporal Euler equation equating the marginal rate of substitution (MRS) of consumption between the two time periods with the marginal rate of transformation (MRT). In a standard model with perfect property rights, the MRT for agent R would simply be A_R; here, because of insecure property rights, it is $p'A_R$. The second condition, (1.7) reflects the equality of marginal returns across different the two types of investment activities. An unit of resource can be invested either in productive or in appropriative activities. In equilibrium, these avenues should generate the same return.

Analogously, the reaction functions for agent P are given by

$$\frac{1}{C_{1P}} = \beta \frac{(X_P)^m}{(X_R)^m + (X_P)^m} A_P \frac{1}{C_{2P}}, \tag{1.8}$$

$$\frac{A_P}{A_R K_R + A_P K_P} = \frac{(X_R)^m}{(X_R)^m + (X_P)^m} \frac{m}{X_P}. \tag{1.9}$$

We can use the symmetry of the reaction functions for the two agents to write $(A_R/A_P) = (X_P/X_R)^{m+1}$ and use in (1.1) to get

$$p' = \frac{1}{1 + \left(\frac{A_R}{A_P}\right)^{\frac{m}{m+1}}}. \tag{1.10}$$

Notice how the appropriation function in (1.1) is transformed to depend solely on the ratio of the technology choices of both agents.

The above formulation of p' highlights the possibility of wealth-ranking reversal in this setup. To see this, suppose the technologies adopted satisfy $A_R > A_P$ (i.e., suppose the initially-wealthier agent adopts the superior technology). Then, (1.10) makes clear that $p' < 1/2$ is possible even when $p > 1/2$ was true. In other words, a wealth-ranking reversal is possible. The fact that there is a scope for redistribution of wealth, from the wealthier and more productive agent to the poorer one, should not come as a surprise. After all, the agent choosing the superior technology has a higher opportunity cost of investing in appropriative activities, which in turn give him a comparative advantage (relative to the other agent) in production. The optimal allocation of saving between different investment activities (or, the equalization of marginal return across productive and appropriative activities) implies that the agent invests more in production and cut back on appropriative investments, and thus end up with less share of future output.

Using (1.6)-(1.10), it is possible to derive the optimal allocation of resources to consumption and appropriation in terms of the stage-one technology choices of both parties. The optimal choices for agent R are given by

$$C_{1R} = \frac{A_P}{A_R}C_{1P} = \frac{\left[(p + (1-p)\frac{A_P}{A_R})Y\right]}{\beta(1+m) + 2}, \tag{1.11}$$

$$X_R = \left(\frac{1}{\left(\frac{A_P}{A_R}\right)^{\frac{m}{m+1}} + 1}\right) \frac{m\beta\left[(p + (1-p)\frac{A_P}{A_R})Y\right]}{2 + \beta(1+m)}, \tag{1.12}$$

and

$$C_{2R} = \frac{\beta}{1 + \left(\frac{A_R}{A_P}\right)^{\frac{m}{m+1}}} \cdot \frac{[(pA_R + (1-p)A_P)Y]}{2 + \beta(1+m)}. \tag{1.13}$$

Analogous expressions for agent P are given by

$$C_{1P} = \frac{A_R}{A_P}C_{1R} = \frac{A_R}{A_P}\frac{\left[(p + (1-p)\frac{A_P}{A_R})Y\right]}{\beta(1+m) + 2}, \tag{1.14}$$

$$X_P = X_R\left(\frac{A_R}{A_P}\right)^{\frac{1}{1+m}} = \left(\frac{A_R}{A_P}\right)^{\frac{1}{1+m}}\left(\frac{1}{\left(\frac{A_P}{A_R}\right)^{\frac{m}{m+1}} + 1}\right)\frac{m\beta\left[(p + (1-p)\frac{A_P}{A_R})Y\right]}{2 + \beta(1+m)}, \tag{1.15}$$

and

$$C_{2P} = \left(\frac{A_R}{A_P}\right)^{\frac{m}{1+m}} C_{2R}. \tag{1.16}$$

If the income distribution is highly unequal, we may end up at a corner solution where the poorer agent does not contribute anything to productive investment and invests only in appropriation. Similarly, the richer agent may have absolute advantage in appropriation. Implicitly then, we assume that the initial distribution of income is not very skewed i.e., p is not very close to 1.

From the expressions of (1.11), (1.13), (1.14), (1.16), it is evident that if the initially-wealthier agent adopts a superior technology, he enjoys less consumption in both periods than the poorer agent. Also note that the equilibrium share of output is less for the relatively more-productive agent. These results are invariant to whether the more-productive agent is initially richer or not. This is because equilibrium allocation of resources are determined by comparative advantage. For example, when $A_R > A_P$, agent R has a comparative advantage in production and poor in appropriation. From standard trade theory, it follows that agent P should invest relatively more in appropriation and thus enjoy higher second-period consumption i.e. $C_{2P} > C_{2R}$. On the other hand, agent P is reluctant to sacrifice current consumption to increase the size of the pie as he is relatively less productive, and therefore, he consumes more in the first period i.e., $C_{1P} > C_{1R}$. Similar arguments hold when $A_R < A_P$.

It remains to incorporate these optimal decisions, (1.11)-(1.16), in the agents' utility max-

imization problem and compute the technology choices (A_R, A_p) in stage one that produce non-cooperative optima for each agent. In other words, we compute U_R as a function of A_R (given A_p) and U_p as a function of A_p (given A_R). These represent the mutual best-responses. A pure strategy Nash equilibrium is a fixed point of these best-response functions that is consistent with positive levels of productive and appropriative investments, and consumption in each period, by both agents.

Proposition 1. *(Gonzalez, 2005) If p is sufficiently close to half and $\frac{A^H}{A^L} \to 1$, then a pure-strategy Nash equilibrium exists. $\left(A_R = A^H, \ A_P = A^H\right)$ is not a pure-strategy Nash equilibrium, i.e., the equilibrium technology profile cannot involve each agent adopting the best available technology.*

Why might agents not wish to adopt the best available technology even when it is costlessly available? In this environment of insecure property rights, the answer lies in the anticipation of future conflict. While adoption of a better technology by an agent raises tomorrow's common output, the very increase in tomorrow's pie elicits a harmful response from his rival (in the form of an increase in appropriative investment), and this dissuades the agent from adopting superior technologies in the first place. More specifically, the optimality conditions imply that agents allocate resources by equating marginal returns from the two types of investment activities. It follows that adoption of a superior technology raises the opportunity cost of appropriative investments for the adopter, inducing him to shift resources from appropriative to productive activities. Ceteris paribus, this raises future common output. On the flip side, the adoption of a superior technology lures his opponents to specialize in appropriation – appropriative investments act as strategic substitutes – thereby increasing the "ex-post tax" on the returns to adoption. The upshot is that choosing to adopt a superior technology confers a strategic disadvantage in the subsequent distribution of wealth.

The starting point of our analysis is this striking result in Gonzalez (2005): people are hesitant to adopt superior technologies because of the fear of subsequent heightened conflict. This presents a prima facie case for some sort of external intervention. Would it help, if a third party, say, a government, intervenes in this conflict by providing some manner of public

protection of rights on private property? In the next section, we take up a slice of this issue.

1.3   Guard posting: introducing public security

1.3.1   Modified environment

To implement the idea discussed above, we introduce a third party, called "government" in the framework of the benchmark model. We think of the government as imposing a non-distortionary tax on the initial endowments of each agent at the start of their life. The tax proceeds are utilized to finance the hiring of a "guard". In terms of the model economy, the guard is simply a public security service whose sole aim is to reduce the effectiveness of each agents' appropriative investments by a constant amount. Since agents' share of future output depends on their effective appropriative investments, the presence of a guard, in effect, creates a threshold below which all appropriative investments are rendered ineffective. This influences agents' decisions on allocation of resources to various activities, which in turn, affects their marginal returns. The question at hand is: can the presence of a guard induce a reallocation of resources in such a way that adoption of the best-available technology by each agent evolves as a Nash equilibrium? [6]

As discussed above, assume each agent is required by law to pay as a tax, a fixed proportion (τ) of his inherited wealth. Since inherited wealth is exogenously-specified – pY for agent R and $(1-p)Y$ for agent P – the tax is non-distortionary. We denote the total tax revenue by G, where $G = \tau Y$. The government uses the tax proceeds to post a guard whose only job is to equally reduce the effective amounts of the appropriative investments of *each* agent. Specifically, if X_i^e is the effective appropriation investment for agent i, then $X_i^e \equiv X_i - G$ where X_i is the corresponding investment made by agent i in the benchmark model. The technology of conflict, the analog of (1.1), is redefined in the following manner:

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[6]By posting a guard, the government can act as a more-effective deterrent against one party capturing more of the final output than is due to that party. A question that legitimately arises at this juncture is, why does the government, via the posting of a guard, get involved in this conflict in the first place? Presumably, the government cares about improving property rights. A fuller discussion of this issue is presented in Section 1.4 below.

$$p'_G = \frac{(X_R^e)^m}{(X_R^e)^m + (X_P^e)^m}. \tag{1.17}$$

The new formulation, which looks a lot like (1.1), maintains the properties of symmetry and homogeneity of degree zero in *effective* appropriative investments; this keeps the model analytically tractable. This formulation requires that each agent invests at least an amount G – the threshold – to get a positive return from appropriative activities. Since τ can be quite small, the threshold – the restriction that $X_i^e > 0$ has to hold – may not be too onerous for the agents. What is important to note is that diminishing returns in appropriative investments imply that the marginal effect of an extra unit invested in appropriation (over and above the threshold) is much lower than in the benchmark model; additionally, the marginal return on appropriative investments is lower than the marginal utility from consumption or the return to productive activities.

It is evident that compared to the benchmark model, the qualitative changes in this section are the imposition of a tax in the first period and the modification of the share function/technology of conflict. The sequence of activities and the information available to each agent at each point of time are exactly the same as that in the baseline model. Therefore, we proceed exactly as before to obtain the set of sub-game perfect Nash equilibria (SPNE).

1.3.2 Equilibrium

Analogous to (1.6)-(1.7), the interior optimality conditions for agent R are given by:

$$\frac{1}{C_{1R}} = \frac{\beta p'_G A_R}{C_{2R}}, \tag{1.18}$$

and

$$\frac{m(X_P - G)^m}{(X_R - G)\{(X_R - G)^m + (X_P - G)^m\}} = \frac{A_R}{Y'}. \tag{1.19}$$

The first condition, (1.18), is the familiar inter-temporal Euler equation that equates the marginal utility of an unit of consumption across periods. For agent R, an unit of consumption forgone today and invested in the productive technology produces A_R units of future output.

Since property rights are insecure, agent R gets to consume only his effective share, $p'_G A_R$. The second optimality condition requires that the marginal returns from both types of investment activities – productive and appropriative – be equated in equilibrium.

It is easy to check that (1.10) continues to hold in this reformulated environment, i.e.,

$$p'_G = \frac{1}{1 + \left(\frac{A_R}{A_P}\right)^{\frac{m}{m+1}}} \tag{1.20}$$

holds. Analogous to (1.11)-(1.16), we now have

$$C_{1R} = \frac{Y\left[\left(p + (1-p)\frac{A_P}{A_R}\right)(1-\tau) - \left(1 + \frac{A_P}{A_R}\right)\tau\right]}{\beta(1+m) + 2}, \tag{1.21}$$

$$C_{2R} = \frac{\beta}{1 + \left(\frac{A_R}{A_P}\right)^{\frac{m}{m+1}}} \cdot \frac{Y\left[(pA_R + (1-p)A_P)(1-\tau) - (A_R + A_P)\tau\right]}{2 + \beta(1+m)}, \tag{1.22}$$

$$C_{1P} = \frac{Y\left[\left(\frac{A_R}{A_P}p + (1-p)\right)(1-\tau) - \left(1 + \frac{A_R}{A_P}\right)\tau\right]}{\beta(1+m) + 2}, \tag{1.23}$$

and

$$C_{2P} = \frac{\beta\left(\frac{A_R}{A_P}\right)^{\frac{m}{m+1}}}{1 + \left(\frac{A_R}{A_P}\right)^{\frac{m}{m+1}}} \cdot \frac{Y\left[(A_R p + (1-p)A_P)(1-\tau) - (A_R + A_P)\tau\right]}{2 + \beta(1+m)}. \tag{1.24}$$

Additionally,

$$X_R = \left(\frac{1}{\left(\frac{A_P}{A_R}\right)^{\frac{m}{m+1}} + 1}\right)\frac{m\beta\Delta}{2 + \beta(1+m)} + \tau Y, \tag{1.25}$$

and

$$X_P = \left(\frac{1}{\left(\frac{A_P}{A_R}\right)^{\frac{m}{m+1}} + 1}\right)\frac{m\beta\Delta}{2 + \beta(1+m)}\left(\frac{A_R}{A_P}\right)^{\frac{1}{m+1}} + \tau Y \tag{1.26}$$

hold where $\Delta \equiv \left[\left(p + (1-p)\frac{A_P}{A_R}\right)(1-\tau)Y - \left(1 + \frac{A_P}{A_R}\right)\tau Y\right]$. It is clear from (1.25)-(1.26) that X_R^e and X_P^e are positive.

What are the main margins on which all the action in this model rests? First, at the margin, a higher tax rate reduces disposable income generating a first order negative effect on

utility. However, there may arise a countervailing positive effect since the proceeds from the tax are used to employ a guard, whose actions may help secure property rights, and thereby encourage better technology adoption. How might this happen? Recall that the presence of a guard creates a threshold below which all appropriative investments are rendered ineffective. As a result, the marginal effect of an extra unit invested in appropriation (over and above the threshold) is considerably lowered, raising the corresponding return from productive activities. Both agents now have an incentive to respond to these favorable returns by adopting better technologies. The whole thing turns on the following tension: does the presence of a guard reduce the anticipation of future conflict by so much that the benefit to agents from adopting superior technologies outweighs their contribution to the financing of the guard in the first place? The next proposition argues that for a range of tax rates, the answer may be in the affirmative.

Proposition 2. *(Guard-posting) If $p \to 1/2$ and $\frac{A^H}{A^L} \to 1$, a pure strategy equilibrium with positive investment in productive activities exists for $\tau \leq \tau_{inv}$. Moreover for $\tau \in [\tau_H, \tau_{inv}]$, $\left(A_R = A^H, \ A_P = A^H\right)$ can be achieved as a Nash equilibrium technology profile.*

The definitions of τ_{inv} and τ_H – all in terms of underlying parameters – can be found in the appendix. Proposition 2 is the central analytical result of our paper. It argues that under the same sorts of parametric restrictions imposed in Proposition 1, a publicly-financed guard can significantly improve the equilibrium technology choice. In particular, $[A^H, A^H]$ can now be supported as a Nash equilibrium, something that was not possible in Proposition 1 or in Gonzalez (2005). [7]

1.3.2.1   Welfare Analysis

As discussed earlier, there is a tension between utility losses from lower disposable income when young and possible welfare gains from superior technology adoption in the presence of

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[7]A few words about Proposition 2 are in order. When the tax rate lies within the interval $[\tau_H, \tau_{inv}]$, each agent's best response is to choose either the best or the worst available technology. That is, any equilibrium technology profile must be situated in the boundaries of the set of available technologies. If the tax rate lies outside the interval $[\tau_H, \tau_{inv}]$ then emergence of an interior equilibrium in technology choice is possible. In the baseline model, this was never a possibility.

a guard. On net, can we say anything about overall welfare levels with and without public provision of security? To that end, we posit a Benthamite social welfare function:

$$SWF \equiv U_R + U_P. \tag{1.27}$$

Since there are multiple equilibria possible both in the benchmark and in the guard-posting models, indeed the set of equilibria are different, the choice of which equilibria to compare becomes critical. Here we choose to compare social welfare across two symmetric equilibria, $\left(A^L, A^L\right)$ in the benchmark model and $\left(A^H, A^H\right)$ in the guard-posting model.

Corollary 1. *If $\left(A^H, A^H\right)$ and $\left(A^L, A^L\right)$ are equilibrium technology profiles in the guard-posting model and the benchmark model respectively, then aggregate social welfare is higher in the former equilibrium if the following parameter condition holds:*

$$\frac{A^H}{A^L} \geq \left(\frac{1}{(1-3\tau)^{2(\beta+1)}}\right)^{\frac{1}{2\beta}}.$$

Before we close this section, it would be useful to summarize our findings thus far. Gonzalez (2005) argued that a primary reason for technological backwardness is insecurity of property rights. If agents anticipate increased conflict from adoption of a superior technology, they may choose not to. The best-available, and yet free, technologies may never be adopted, with serious consequences for growth and welfare. We introduced the notion of public security of private property rights. In our setup, a guard is posted by the government with the sole aim of reducing the effectiveness of the appropriative investments of each agent. We find that the best-available technology can now be supported as a Nash equilibrium. This new equilibrium may also exhibit superior welfare.

In the environment studied thus far, the extent of involvement of the government in the post-production conflict was limited to posting a guard. All the guard did was thwart the appropriative activities of each agent, much like a policeman would. As an intuition-building exercise, this thought experiment was useful. What happens if the government takes on a more direct, proactive role in the post-production conflict, and is not restricted to merely impeding

the appropriative activities of agents?

1.4 Improving property rights

In this section, we allow the government to utilize the tax proceeds to directly influence the technology of conflict with a view to improving the security of private property rights. This is achieved via the following reformulation of the conflict technology:

$$p'_e = \frac{x_R^{m\theta}(A_R K_R)^{1-\theta}}{x_P^{m\theta}(A_P K_P)^{1-\theta} + x_R^{m\theta}(A_R K_R)^{1-\theta}}, \ \theta \in [0,1]. \tag{1.28}$$

In this formulation, p'_e denotes the share of second-period output that accrues to agent R. As is clear from (1.28), p'_e reduces to p' (see (1.1) in the benchmark model) when $\theta = 1$ and to $A_R K_R / Y'$ when $\theta = 0$. In other words, the technology of conflict in (1.28) straddles two extremes, the insecure property-rights regime from the benchmark model and an environment of perfect property rights (where agent R receives his legitimate share, $A_R K_R / Y'$).

We posit that θ is a choice variable for the government albeit not a costless choice. Real resources are diverted to enhance property rights. Specifically, the government can influence θ directly by spending G where $\theta \equiv \Phi(G)$, and $G = \tau Y$. Furthermore, $\Phi(0) = 1$, $\Phi(G^*) = 0$, and $\Omega'(G) < 0$. If the government wishes to improve property rights, it raises τ (and hence, G) and uses the revenue to reduce θ.[8] In the limit, as G approaches a critical level, G^*, a perfect property rights regime is established. In a laissez-faire regime, the government takes no part in post-production conflict and sets $G = 0$. This establishes the polar opposite regime of insecure property rights. Henceforth, θ measures the exact level of insecurity of agents' claims to private property.

The rest of the environment is exactly as it is in the benchmark model. Analogous to

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[8]This action could be interpreted as improving funding for the police and the judiciary at large.

(1.6)-(1.7), the interior optimality conditions for agents R and P are given by

$$\frac{1}{(1-\tau)pY - X_R - K_R} = \frac{\beta A_R}{A_R K_R + A_P K_P} + \frac{(1-\theta)X_P^{m\theta}(A_P K_P)^{1-\theta}}{[X_P^{m\theta}(A_P K_P)^{1-\theta} + X_R^{m\theta}(A_R K_R)^{1-\theta}]K_R} \quad (1.29)$$

$$\frac{1}{(1-\tau)pY - X_R - K_R} = \frac{m\theta X_P^{m\theta}(A_P K_P)^{1-\theta}}{[X_P^{m\theta}(A_P K_P)^{1-\theta} + X_R^{m\theta}(A_R K_R)^{1-\theta}]X_R}, \quad (1.30)$$

and

$$\frac{1}{(1-p)(1-\tau)Y - X_P - K_P} = \frac{\beta A_P}{A_R K_R + A_P K_P} + \frac{(1-\theta)X_R^{m\theta}(A_R K_R)^{1-\theta}}{[X_P^{m\theta}(A_P K_P)^{1-\theta} + X_R^{m\theta}(A_R K_R)^{1-\theta}]K_P}$$

$$(1.31)$$

$$\frac{1}{(1-p)(1-\tau)Y - X_P - K_P} = \frac{m\theta X_R^{m\theta}(A_R K_R)^{1-\theta}}{[X_P^{m\theta}(A_P K_P)^{1-\theta} + X_R^{m\theta}(A_R K_R)^{1-\theta}]X_P}, \quad (1.32)$$

respectively. The equilibrium technology profile involves solving the above system of equations – (1.29)-(1.32) – for K_R, K_P, X_R and X_P, where $p \in (1/2, 1]$, $\beta \in [0, 1]$, $\tau \in [0, 1]$, $m \in [0, 1]$, $\theta \equiv \Phi(G) \in [0, 1]$, and $Y > 0$. The nature of non-linearity in the system of equations severely restricts the scope for analytical solutions. We resort to a numerical analysis.

1.4.1 Numerical Analysis

The model economy, and hence, the system eqs. (1.29)-(1.32), has undergone a substantial change over the model described in Section 1.2. There, as Proposition 1 had established, all Nash equilibria lay at the boundary of the available technologies set, i.e., either A_P or A_R could take the boundary values A^H or A^L but not an interior value. No such guarantees are available to us in the system, (1.29)-(1.32). Multiple, possibly interior, equilibria are clearly possible here. Since we are ultimately interested in studying changes in θ, matters could get tricky especially if a change in θ takes us from one equilibrium to another. To keep the analysis in this section comparable with Sections 1.2 and 1.3 below, we will restrict the analysis to a single Nash equilibrium, $\left(A_R = A^H, \ A_P = A^H\right)$, even though many others, possibly even a continuum, are possible. Within the confines of this single equilibrium, the one corresponding to both parties choosing the frontier technologies, we will ask, how do various variables of interest vary as θ changes? Specifically, as θ falls (i.e., property rights become more secure),

how does growth, inequality, and welfare respond? The question uppermost on our mind is, is government-funded increased security of property rights a good idea always?

We develop the following numerical scheme in order to simulate the system, eqs. (1.29)-(1.32), so as to analyze the effect of property rights improvement (through effective government intervention) on relevant choice variables. Since the model economy is quite stylized, the numerical exercise below is not to be understood as a calibration exercise, rather the exploration of a particular equilibrium using numerical methods.

We begin by specifying the values of the parameters, the range of tax rates, and the set from which the technology is chosen: $\tau \in (0, 20\%)$, $Y = 100$, $\beta = 0.8$, $A_i \in [A^L = 18.9, A^H = 20]$, $m = 0.5$, $p = 0.6$, $\Phi(G) \equiv 1 - \frac{(G)^\alpha}{K^\alpha}$ with $K =$, and $\alpha = 0.5$. To stay in line with Propositions 1-2, we choose $(A^H/A^L) \approx 1$ and p close to $1/2$. Clearly, Y and β are scale parameters and are easily varied without any change in the qualitative properties. The tax rate is kept in a reasonable range of under 20% (indeed, much of the action below happens for tax rates below 10%). α represents the elasticity of effective property rights with respect to government spending.

In steps 1 and 2 below, we summarize the algorithm that we use to identify the set of tax rates that supports the choice of best available technologies as a Nash equilibrium for the poor agent. A similar scheme is developed for the rich agent. Finally, in step 3, we find the interval of tax rates that supports the choice of frontier/best-available technologies for both agents as a Nash equilibrium.

1. **Step** 1 : We start by making a grid for τ (the tax rate) and A_P (the technology choice of the poor agent). Given an initial choice of τ at the first grid point, we perform the following analysis: We fix A_R at its highest possible level, $A_R = A^H = 20$, and choose the first grid point of $A_P = A^L = 18.9$. For the given choice of parameters, we simultaneously solve the above-discussed system of non-linear equations (using the Matlab in-built function "simulnonlinear") to get the optimal values of C_{1R}, C_{1P}, X_P and X_R. Using these, we compute K_R, K_P, C_{2R}, and C_{2P}. Next, we evaluate indirect utility of agent P, $U_P \equiv \ln C_{1P} + \beta \ln C_{2P}$, which depends on the initial choice of τ,

$A^H = 20$ and $A_P = 18.9$. Keeping the initial choice of τ unchanged, we change A_P along the grid, holding A_R fixed at A^H to see how U_P changes with A_P. This process is iterated twenty times. For the initial choice of τ, the indirect utility curve is plotted as a function of A_P for every iteration.

2. **Step** 2 : To check whether agent P has an incentive to choose the best possible technology when agent R has done so, we compute $\partial U_P / \partial A_P$ at $A_P = A_R = A^H$ for this choice of τ. If the slope is positive, we assert that this tax rate supports the best available technology adoption for the poor and consequently, record the value of τ. A negative slope implies an incentive on the part of agent P to deviate from the best technology choice given agent R has chosen it. In that case, we reject that value of τ and proceed to repeat the same exercise for the next point on the grid. This process is repeated for the entire grid of τ and record those τ for which the aforementioned slope is positive. Denote this set by $S_1 = [\tau_1, \tau_2]$.

3. **Step 3:** An analogous exercise is performed for agent R and a set $S_2 = [\tau_3, \tau_4]$ is found. We denote $S = S_1 \cap S_2$ as the range of tax rates that supports $\{A^H, A^H\}$ as the Nash equilibrium. In our case, $S = [0.0020, 0.1001]$.

The graphical representations summarizes the central findings of the numerical analysis. An intuitive explanation of the underlying dynamics is presented subsequently, in the discussion section.
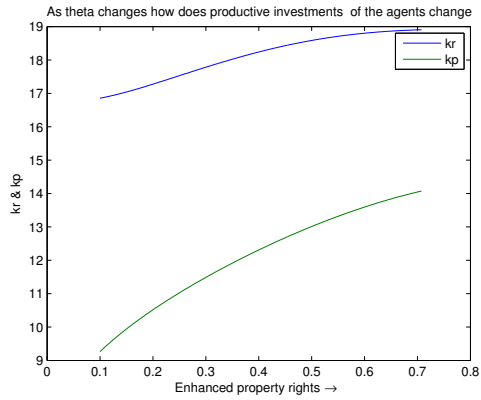
Productive Investments
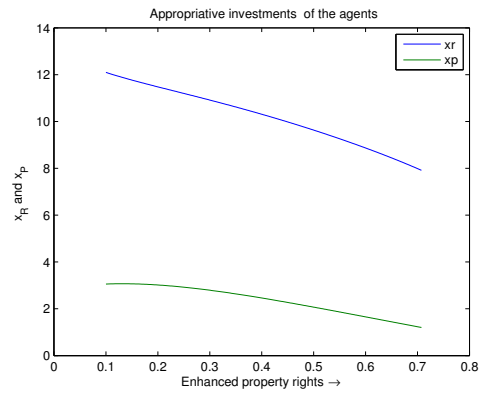
Appropriative investments
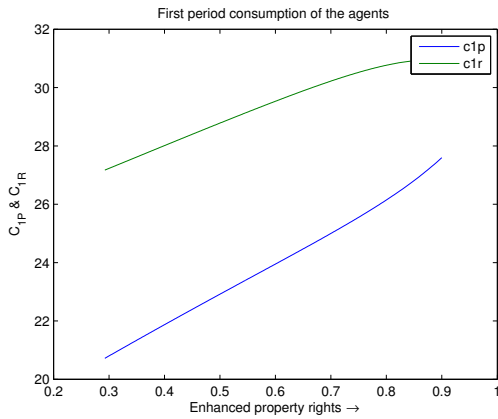
Current consumptions

Future consumptions

Growth in Output

Aggregate welfare

Percentage change in property rights

Indirect Utility of Rich

Indirect Utility of Poor

Elasticity w.r.t. optimal property rights

Effect of distribution of Income on SWF

1.4.2   Discussion

Here, we study how an improvement in property rights shapes optimal resource allocations when the best-available technology has already been adopted. Intuition suggests that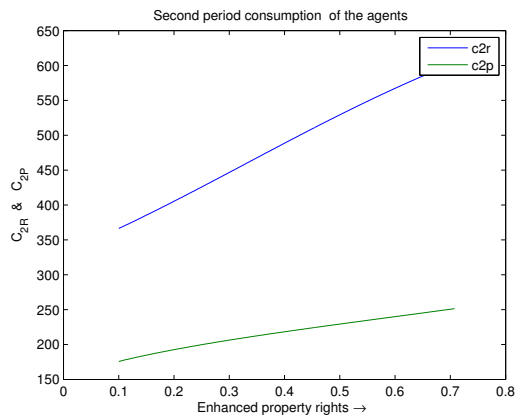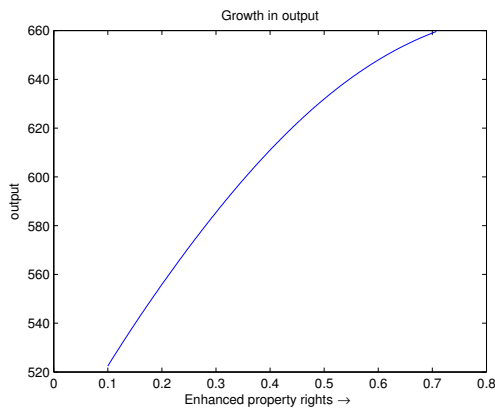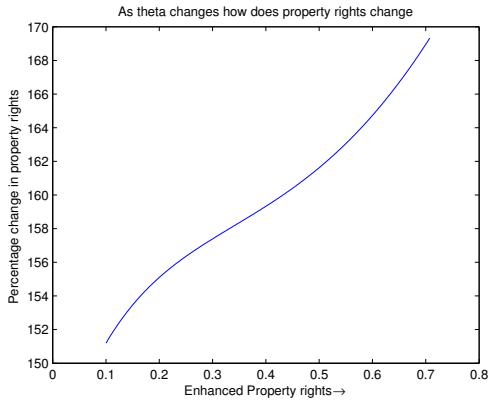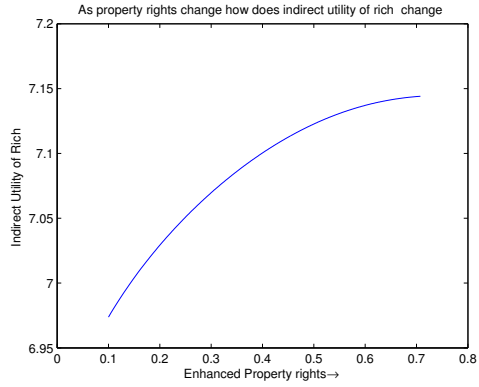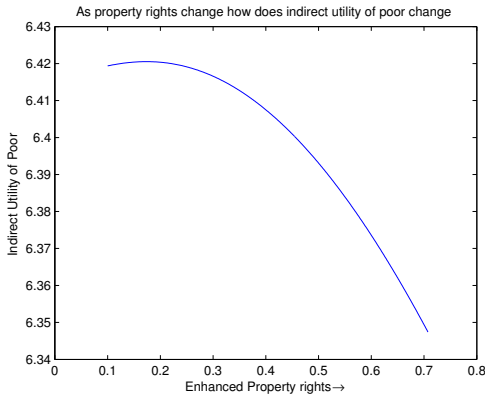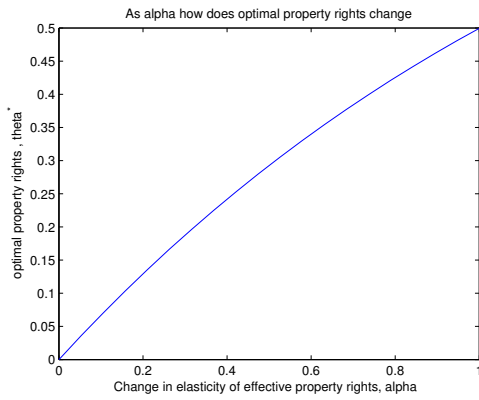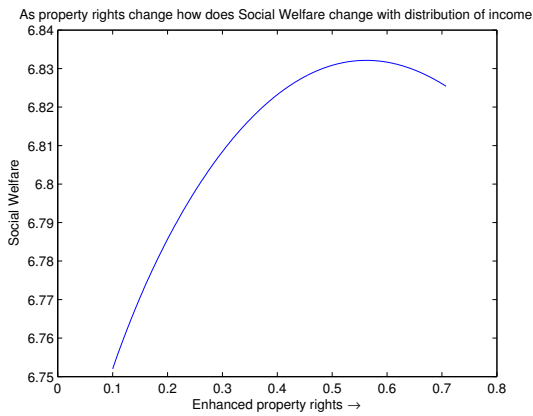 within this interval, enhanced security of property rights should induce larger productive investments and thereby foster economic growth. Would this benefit come at the cost of lower welfare? Are more secure property rights always desirable? If the government could ensure perfectly secure property rights, would it?

The figures above summarize the movements in resource allocation and other important economic indicators with improvements in property rights. When $1-\theta$ increases, property rights improve, productive investment for both the agents go up and appropriative activities fall. It is evident from the figures that agents sacrifice first period consumption along with appropriative investment and allocate the resources towards productive investment in an anticipation of higher second-period consumption. However, the *rates* at which these changes occur varies significantly across the two agents. Growth of output shows a steady positive relationship with improvement in property rights, which can be logically concluded from the effect on capital accumulation. However, effect on social welfare is non-monotone.

The economic rationale behind the movements in the resource allocation is intuitive and foreseeable from the nature of the problem and the framework considered. The central result here is the growth-welfare trade-off. An improvement in property rights and institutional arrangement induces a reallocation of resources towards productive investment at the expense of appropriation and first period consumption. These effects can be traced back through different avenues for both the individuals and the reason can be attributed to their initial comparative advantage and individual resource allocations. For the poor agent, who had a comparative advantage in appropriation, return from productive investment increases unambiguously which in turn shifts resources towards production form appropriation and current consumption. For the initial rich, this reallocation of resources results from a decrease in the return from appropriation. The effect of improved property rights on the return from production for the rich agent is ambiguous. However, it is insightful to note that, this is a composite of a direct effect

of θ and an indirect effect of θ on p' on the return from the productive activities. Similar effects work for the return from appropriation for the poor agent. From the figures, it is clear that investment in production remains the more lucrative option although the rate of capital accumulation varies significantly with improvement in property rights. This is apparent from the curvature of the capital investment curve which is initially increasing and convex and thereafter, concavity sets in. This concavity is a result of diminishing marginal return, which plays a crucial role in explaining the growth welfare trade-off.

In a Benthamite definition of welfare, both agents are treated equally. In our framework, that simply means the direction of movement in welfare results from the interaction between current and future consumption. Initially, when property rights improves, agents sacrifice appropriative investments and first period consumption in an anticipation of increased second period consumption. Since capital is accumulated at a rapid pace, this anticipation is fulfilled and consequently, we obtain an increasing trend in the welfare. This process continues until we reach a critical value of property right parameter θ, beyond which increase in future consumption is outweighed by the fall in current consumption. This effect can be justified along the following lines: after significant amount of productive investment is undertaken, diminishing returns set in. Though agents still keep reallocating their resources towards production, the increase in second period consumption fail to dominate the loss in utility and thus welfare begins to exhibit a steep decline. In other words, enhanced security of private property promotes productive investment and thus fosters economic growth, which might necessitate a sacrifice in current consumption. The trade off between growth and welfare is central to the policy analysis of this research. An improved property right environment (generated by effective government intervention) that fosters economic growth might not be optimal from a welfare point of view.

1.5   Conclusion

We have considered the role institutions of property rights and conflict management can play in both achieving prosperity and mitigating conflict in developing countries. In the first half of our paper, we consider a scenario where public-funded protection of private property rights may successfully support the adoption of best-available technologies as Nash equilibrium.

Such a scheme may even be welfare enhancing. Here the government's role in post production conflict is limited to " posting a guard" who thwarts the appropriative activities each agent much like a policeman Next we try to answer a more pertinent question: what happens if government takes on a more direct, proactive role in post-production conflict? Basically we endogenize the property rights by introducing a new formulation of the conflict technology, where government can explicitly intervene in the existing level of property rights by choosing the tax rate. We allow the government to utilize the tax proceeds to directly influence the technology of conflict with a view to improve the security of private property. With in this set up, we study how an improvement in property rights shapes optimal resource allocations when the best-available technology has already been adopted. This addresses a fundamental question. When the government has the option to choose a tax rate that ensure perfect property rights, is that always desirable? Would such a choice of tax rate be always welfare enhancing? We show that there exists an interval of taxation such that an increase in property security leads to a decrease in welfare. From a policy perspective this surprising result calls for a caution in recommending improved property rights enforcement, particularly when such improvements are to be made incrementally in middle-income countries.

CHAPTER 2.   Stock Loans Subject to Bankruptcy

2.1   Introduction

A stock loan or a security loan is a security based lending arrangement where an owner of an asset can borrow money from a lender using the asset as a collateral. The terms of the loan are governed by a "securities lending agreement", which requires that the borrower provides the lender with collateral, in the form of cash, government securities, or a letter of credit of value equal to or greater than the loaned securities. By construction, the stock loan provides the borrower with the opportunities to create liquidity today from their equity position without selling and hedge downside risk. Moreover, this financial instrument also minimize the impact of short term volatility and thus provide significant potential profit from future stock appreciation (each of the concepts will be explained subsequently). For the specific reasons, the stock loan has become a very popular financial instrument and according to the industry group ISLA, in the year 2007 the balance of securities on loan globally exceeded 1 trillion.[1] In the context of a less developed country perspective, high degree of volatility of asset prices makes such a security based arrangement more risky as well as lucrative at the same time. A significant unutilized potential and vulnerability to the new information makes the asset prices much more volatile, making the possibility of financial gain wider. On the flip side, high degree of volatility also makes default events and bad loans more probable. In this paper, we incorporate the risk of bankruptcy in the valuation of stock loans to give the analysis a realistic flavor as well as to widen applicability from the policy perspective.

To fix the idea of stock loans in a model economy, let us consider an economy consisting of two entities, a borrower (the client) and a lender (a bank). The investment options in this economy

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[1]Please refer to the article on "Securities lending" in Wikipedia, The Free Encyclopedia and the reference therein of "An introduction to securities lending", 1 August, 2005, Executive Summary, Page 8.

are limited to a risky asset (a stock), and a safe asset (a bond or, equivalently, a bank account). The client owns one share of the stock as an asset and obtains a loan from the bank using the share as a collateral. The bank charges the lending interest rate on the principal. The terms of the contract are such that the agent may choose to regain the stock at any point of time by paying the principal and accumulated interest or else, he may choose to surrender the stock, instead of repaying the loan. Such a loan is called a stock loan or a security loan. A stock loan, by construction, allows the client to achieve a variety of objectives. Most importantly, it serves as a hedge against market downturn. As it is evident from the construction, a rational agent will choose to regain the stock only if the stock price exceeds the principal and the accumulated interest. In any other scenario, the best response of the client is to forfeit the stock instead of repaying the loan. Therefore, maximum loss incurred by the client is the initial price of the stock less the loan principal. On the other hand, the client has unlimited potential of gaining money, resulting from a significant increase in the stock price. If the stock price goes up, the client repays the principal plus the interest and keeps all the surplus. At this point, it is illuminating to mention the analogy of the model scenario with real world mortgage finance. Investing money in the purchase of a durable asset (for example, a house) financed by mortgage payments essentially gives rise to similar situation when the calculated payments are made on a monthly basis for a stipulated period of time. At any point, the borrower can decide to stop the stream of payment and the financial institution seizes the borrower's ownership right over the asset forever. Thus we note that the framework considered here is quite general and can represent real world mortgage financial issues with minor modification.

The problem of stock loans, precluding the risk of bankruptcy, resembles to an option valuation of the American type. From the general equilibrium perspective, it is an optimization problem from both lenders and borrowers point of view. To elaborate, lender's problem is to maximize the expected profit subject to the incentive compatibility and individual rationality constraints. Also, in a model with more than one agents, the problem of adverse selection needs to be incorporated in the lender's optimization problem. In this simplified framework, the lender has to decide the optimal amount to lend so that it maximizes the expected payoff.

The optimization problem faced by the lender is out of the scope of this research and therefore, not addressed here. Problem faced by a rational borrower is crucial in the valuation of such a financial instrument. Payoff accrued to borrower is simply the difference between the current stock price and interest accumulated loan amount. Though the payoff is positively related with the stock price, waiting has a cost in terms of accumulated interest payment. Facing the tradeoff, the most crucial question from the client's point of view is when is the optimal time to regain the stock so as to maximize the present value of the discounted expected payoff. In a partial equilibrium analysis, where the loan amount is taken as given to the agent, the problem is to find the optimal time to exercise the option. A reasonable characterization of the optimal time require that agent is cognizant and well informed about the decision to exercise when the time arrives. In mathematical language, the problem is to find *an optimal stopping time*. In a stochastic environment, where stock price is modeled as a stochastic process, this problem indeed is to find an optimal exercise boundary. Intuitively, at every point in time, the agent is faced with a choice, whether to exercise the option or to wait a period longer. Waiting has its cost in terms of accumulated interest payment, coupled with expected future gain from appreciation of asset price. The problem is essentially to find an optimal stopping time to exercise the financial instrument so as to maximize the present value of discounted expected payoff. Valuation of stock loans without the risk of bankruptcy is addressed in Xia and Jhou (2007).[2]

In this paper, the valuation of stock loan is addressed when the underlying stock is subject to risk of bankruptcy. Bankruptcy is a legally declared inability or impairment of ability of an individual or organization to pay its creditors. In other words, it is the condition of a legal entity that does not have the financial means to pay their incurred debts. Modeling asset prices as stochastic processes addresses issue of realism of the research and to this end, introducing risk of bankruptcy is a realistic generalization. Highly volatile assets are sensitive to the economic and political perturbations which in turn causes significant unprecedented fluctuations in asset prices. Asset prices, such as stock or house prices, may very well be subject to the

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[2]In their formulation, the problem reduces to evaluating an American call option with a time dependent strike price. It turns out that the aforementioned problem is not a straightforward adaptation of its constant strike price counterpart. The authors use a probabilistic approach to solve the valuation problem.

risk of bankruptcy or sudden devaluation due to market downturn. A model that precludes bankruptcy, by construction, rules out the possibilities of such events in deriving the optimal stopping time and thus deficient in terms of practical applicability. At the same time, modeling bankruptcy is at the heart of entire credit risk literature which has wide practical applications. A point is worth mentioning at this point. Having the option to buy or exercise but not the obligation is a defining characteristic of financial options and absolutely uncorrelated to the event of bankruptcy. This distinction will be much more transparent upon the introduction of the details of bankruptcy modeling.

In credit risk literature, there are two fundamental approaches to introduce the risk of default in pricing and hedging financial securities. These two approaches differ basically in terms of modeling the information available at the different point in time. These models are often viewed as competing (see Bielecki and Rutkowski, 2002; Rogers, 1999; Lando, 2003; Duffie, 2003), and there are differences in opinion among the professionals and academicians regarding the supremacy of one over another (see Jarrow et al., 2003, and references therein). Before getting into qualitative difference and comparison, let us first present a brief overview and defining characteristics of both the modeling approaches. The first approach originated from the seminal work of Black and Scholes (1973) and Merton (1974), which is broadly termed as *structural approach*. In the original formulation of Merton (1974), default of a bond happens only if the borrower is unable to pay the debt in full at the time of maturity. Clearly, in Merton's framework, default can only happen at the time of maturity. A basic extension of the Merton (1974) model is due to Black and Cox (1976), where the model allows event of default to occur prior to the maturity of the bond. The central idea governing the event of bankruptcy in this formulation is, the asset is declared to be bankrupt the first time asset price hits a pre-determined lower boundary. The boundary, which is a common knowledge between both the parties, is modeled as a deterministic function of time. In modeling the lower boundary, some functional form has been preferred over others mainly because of the analytical convenience, which will be explained in greater detail in the model section. This approach assumes complete knowledge of a very detailed information set and the default event is modeled as a

hitting time of a predetermined barrier. Consequently, the information assumption implies that firm's default time is predictable. Nonetheless, this approach is intuitive, relies mostly on the framework of the celebrated Black and Scholes (1973) analysis and hence has wide applicability. In the first part of our research, we compute the valuation of stock loan subject to the risk of bankruptcy according to the Black-Cox (1976) specification.

The structural form modeling approach, though intuitive and widely applicable, face severe criticism on the grounds of predictability. The second approach, reduced form models, introduces bankruptcy as a non-predictable or inaccessible phenomenon. Reduced form models originated with Jarrow and Turnbull (1992). Subsequently, Jarrow, Lando and Turnbull (1995), Duffie and Singleton (1999), and Hull and White (2000) present detailed explanations of several well known reduced-form modeling approaches. According to Jarrow and Protter (2004), key distinction between structural and reduced form models is not in the characteristic of the default time (predictable versus inaccessible), but in the information set available to the modeler.This approach assumes the event of default is driven by a default intensity that is a decreasing function of the underlying stock price. The intensity function measures the conditional probability of default in an infinitesimal amount of time. The intuition behind modeling of the intensity function can be put forward as higher the stock price is less probable the stock is to default. The agent has access to the information that is revealed by observation of the stock price. The stock is declared to be bankrupt at the first time accumulated conditional probability of default exceeds the realization of an exponential random variable, which is completely uncorrelated and independent to the stock price behavior. Reduced form models are mathematically rigorous and involved and therefore will be explained in a more elaborated manner in model section. Moreover, real time flavors of the actual events of bankruptcy in the sense that, the event of bankruptcy cannot be foreseen given the knowledge of history of the stock price movement. In the second part of the paper, we would evaluate the stock loan problem subject to bankruptcy according to the reduced form specification.

The chapter is organized as follows: The next section introduces the basic stock loan prob-

lem (due to Xia and Jhou, 2007). In section 3, the risk of bankruptcy is introduced in the valuation of the stock loan problem following the Black and Cox specification, and the closed form expression of the valuation is obtained. In section 4, we introduce bankruptcy in reduced form approach and prove the existence of the optimal exercise boundary. Section 5 presents the numerical analysis of parameter sensitivity of the optimal exercise boundary and discuss some related policy issues. Section 6 concludes the chapter and discuss the scope of future research.

2.2 Stock Loan Problem

We consider the standard Black-Scholes model in a continuous time, where model economy consists of two assets: a risky asset (Stock), and a risk less asset (a bond). The continuous compounding risk neutral interest rate is assumed to be a constant at $r > 0$. We consider a filtered risk neutral probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ and let $(\mathcal{F}(t))_{t \geq 0}$ be the associated filtration. Let $W(t)$ be a Brownian motion defined on this probability space and relative to this filtration $(\mathcal{F}(t))_{t \geq 0}$ such that , $W(t)$ is $\mathcal{F}(t)$ measurable for every t and for every $u > t$, the Brownian increment $W(u) - W(t)$ is independent of $\mathcal{F}(t)$. In other words, the information revealed by the Brownian motion at every instant t is captured by the Brownian filtration $\mathcal{F}(t), t \geq 0$, with $\mathcal{F}_0 = \sigma(\phi, \Omega)$ and $\mathcal{F} = \sigma(\cup_{t \geq 0} \mathcal{F}_t)$. As in the celebrated Black-Scholes model, the stock price is assumed to follow the standard model of a geometric Brownian motion (GBM)- a diffusion process with constant drift, $r > 0$ and volatility $\sigma > 0$ with an infinite life time:

$$S_t = S_0 exp\{(r - \frac{\sigma^2}{2})t + \sigma W_t\}, t \geq 0 \tag{2.1}$$

where, $S_0 > 0$ is the initial stock price and $\sigma > 0$ is the implied volatility. For the sake of simplicity, we assume dividend to be zero.

A stock loan contract is a security based lending agreement between the lender (bank) and the borrower (client), where the client, who owns a share of the stock, borrows $q > 0$ amount of money from the bank using the stock position as a collateral. In our partial equilibrium framework, we assume that value of q is optimally decided by the lending authority

and therefore, given to the borrower. Continuous compounding loan interest rate being γ, the client may choose to regain the stock at any time instant $t > 0$ by repaying $qe^{\gamma t}$ to the lending authority. The client has the choice but not the obligation to repay the loan and regain the stock, and hence, he might choose to surrender the stock for ever. At this point, it is instructive to recapitulate the definition of a traditional American call option, in order to make the comparison between the above discussed financial instruments self-explanatory. An *American call option* is a financial contract between two parties where the buyer of the option has the right but not the obligation to buy the agreed quantity of a financial instrument from the seller *any time* before the expiration date for a predetermined price, called *strike price*. The seller is obligated to sell the commodity if the buyer decides to purchase it. From the discussion thus far, it is quite clear that stock loan contract is equivalent to an American type option with time varying strike price, where the payoff process is $Y_t = [S_t - qe^{\gamma t}]^+ = max[(S_t - qe^{\gamma t}), 0]$. The agent implicitly pays $(S_0 - q)$ to buy the option at time zero. In a partial equilibrium framework, the problem faced by the borrower is to find an optimal time to regain the stock so as to maximize the expected net present value of the lifetime payoff. The important characteristic of the exercise time is with the information revealed by the realized stock price, agent should be cognizant and informed about the event of exercise when the time arrives. Mathematically, the optimal exercise time should be a $(\mathcal{F}_t)$-stopping time and this type of optimization problem is broadly termed as an optimal stopping problem. The present value of the discounted expected payoff at the optimal stopping time is exactly the value of the financial instrument. Thus, the problem essentially is to evaluate an American option with time dependent strike price $qe^{\gamma t}$. We call the value of the option *the initial value* or *the price of stock loan*. Technically, starting at time zero (and therefore with the knowledge of initial stock price S_0 only), the problem is to choose a stopping time (from the class of all $(\mathcal{F}_t)$-stopping times) that maximize the value of the option.

The value function of the option, as a function of initial stock price $S_0 = x$ can be written as:

$$f(x) = \sup_{\tau} E[e^{-r\tau}(S_\tau - qe^{\gamma\tau})^+ | S_0 = x] \qquad (2.2)$$

where, τ has to be chosen from all $(\mathcal{F}_t)_{t \geq 0}$ stopping times, and the expectation is taken with respect to the risk neutral probability measure $\mathbb{Q}$. A simple transformation can be used to turn the problem into one with a time-independent strike price.

$$f(x) = \sup_{\tau} E[e^{-(r-\gamma)\tau}(e^{-\gamma\tau}S_\tau - q)^+] \tag{2.3}$$

where, the transformed stock price $\tilde{S}_t \ (= e^{-\gamma\tau}S_t)$ follows a transformed GBM given by:

$$\tilde{S}_t = \tilde{S}_0 \exp\{(r - \gamma - \frac{\sigma^2}{2})t + \sigma W_t\}, t \geq 0 \tag{2.4}$$

In the process of transforming the problem, the effective discount factor $(r - \gamma)$ becomes negative (since the lending interest rate is usually higher than the risk neutral interest rate), which poses a serious problem on the valuation of the option. It turns out that the standard techniques of solving such an optimal stopping problem using *the variational inequalites in differential equations and the smooth-fit principle* does not work because of the negative discount factor. For this specific reason, the valuation of the stock loan is computed using a pure probabilistic approach (Xia nd Jhou, 2004).[3] In their analysis, they have shown that, the optimal stopping time for the problem happen to be:

$$\tau^* = \inf\{t \geq 0 : S_t - qe^{\gamma t} \geq V_t\} \tag{2.5}$$

where, the threshold value function, V_t takes the form of a *snell's envelope.*

$$V_t = ess \sup_{\tau} E[e^{-r(\tau-t)}(S_\tau - qe^{\gamma\tau})^+|\mathcal{F}_t]. \tag{2.6}$$

In a nutshell, the stock loan problem itself is a non trivial extension of American type call option valuation. Introducing the risk of bankruptcy is a realistic generalization which makes the problem widely applicable and mathematically more challenging at the same time. Moreover, in the face of recent severe economic downturn (which have seen a number of well

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[3]Please refer to the original Stock loan paper by Xia and Jhou.

established and reputed corporate houses go bankrupt), risk of bankruptcy while pricing a financial derivative is ubiquitous.

2.3 Introducing Bankruptcy: Structural Form

Structural form models emanated from the pioneering work of Black and Scholes (1973) and Merton (1974). In the original construction of Merton (1974), a firm has a single liability with promised terminal payoff L, interpreted as the zero-coupon bond with maturity T and face value $L > 0$. The ability of the firm to redeem its debt is determined by the total value of firm's assets at time T, V_T . Default may occur at maturity time T only, and the default event corresponds to the event $\{V_t < L\}$. By construction, Merton's model does not allow for a premature default, in the sense that the default is not modeled to occur before the maturity of the claim. Several authors put forward structural-type models in which this restrictive and unrealistic feature is relaxed. Black and Cox (1976) extended Merton's research in valuation of corporate zero coupon bond by taking several specific features of real life debt constraints into account. A salient feature of Black and Cox (1976) model is introduction of the possibility of premature default. We introduce the risk of bankruptcy in the valuation of stock loan following the Black-Cox (1976) specification. Though we work with the basic stock loan set-up, in order to introduce the aforementioned notion of bankruptcy, we need to modify several elements of the model according to the Black-Cox specification. The elements of the modified stock loan problem is documented as follows:

Stock Loan. As in the basic stock loan problem, we consider a simple world with two assets, a stock and a bond. An agent (borrower), who owns one share of a stock, obtains a loan q from the bank (lender) at $t = 0$ with the share as collateral. Lending interest rate fixed by the lender is assumed to be γ. We assume the stock price process follows the same GBM process as in the original stock loan model, given by equation (1).

Maturity and Repayment options. Following the Black-Cox model, we consider the problem in a finite-time horizon. We assume that the loan contract matures at T. At the maturity, the owner has the option but not the obligation to repay the principal with the

interest (or, interest accumulated loan amount) $qe^{\gamma T}$ and regain the stock. However, if the owner chooses not to regain the stock, by default the stock belongs to the lender and owner loses the right on the asset.

Default Barrier. We introduce the notion of bankruptcy following Black-Cox (1976) as follows: within the time period $[0, T]$, the asset is declared to be bankrupt the first time asset price hits a lower boundary. The boundary is modeled as a deterministic function of time. Following Black and Cox (1976), default boundary is taken as:

$$C(t) = C_1 e^{-\delta(T-t)}, t \geq 0. \tag{2.7}$$

In the original formulation of Black and Cox (1976), the boundary represents the point at which *bond safety covenants* cause a default. The primary reason to choose the above functional form for default boundary is analytical convenience. First passage times for diffusions have been thoroughly/rigorously studied and closed form solutions are available in the case where Brownian motion is hitting a linear boundary. To allow for the movements of stock price to analyze bankruptcy, we assume

$$S_0 > C_1 e^{-\delta T}$$

which essentially means at the time of borrowing, stock price exceeds the value of the default boundary. Under this formulation, the event of bankruptcy is modeled as the first time stock price hits the above specified boundary, introduced as following *stopping time*:

$$\tau = \inf\{t \geq 0 : S_t \leq C_1 e^{-\delta(T-t)}\} \tag{2.8}$$

Essentially, τ is the first passage time of the stock-price diffusion process to the specified boundary. A graphical representation of default will be illuminating.

Figure 2.1   Bankruptcy: Black and Cox (1976)
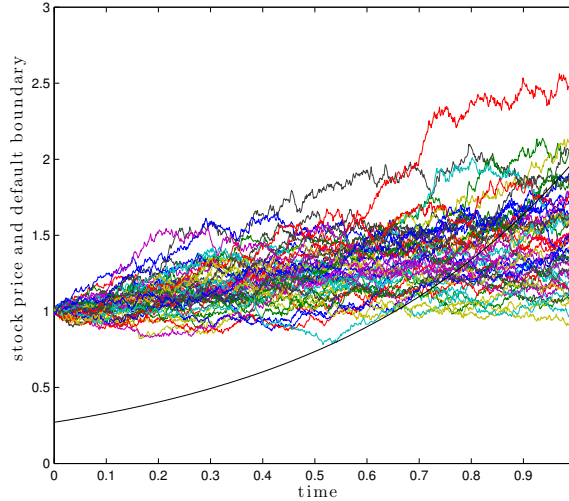
The continuously differentiable increasing curve represents the boundary. Each random graph represents a realization of the stock price process. This formulation essentially means, if the agent is on a path that intersects the boundary at a $\tau \in [0, T]$, then the stock is declared bankrupt. The payment that the client makes to the bank is crucially contingent on occurrence of τ within the interval $[0, T]$. This issue is crucial for the valuation and therefore, addressed in detail in the next subsection.

2.3.1   Payment Structure

In the event of bankruptcy (i.e., if $\tau \in [0, T]$), the stock is declared to be bankrupt and the client no longer have any right on the stock. The bank owns the stock following the terms of the contract and therefore, have the right to liquidate it. Hence, in the event of bankruptcy occurring before the maturity, the implicit payment to the lender is:

$$e^{-r\tau} S_\tau = e^{-r\tau} C_1 e^{-\delta(T-\tau)}$$

In the case of no default prior to maturity (i.e., if $\tau > T$), then the borrower has the right, but not the obligation, to pay $qe^{\gamma T}$ at $t = T$ and get the stock back. The client is not obliged to

regain the stock. Therefore, in case the client does not claim, bank owns the stock and have the right to liquidate it at T. Therefore, in the event of nonoccurrence of the bankruptcy before maturity, the value of the contract to the bank is $Min\{S_T, qe^{\gamma T}\}$.

2.3.2 Valuation

From the theory of option pricing, the value (or price) of the contract at $t = 0$ turned out to be the risk neutral expectation of the present value of the future promised payments.[4]

Denoting this value as V_0, we have:

$$V_0 = E[e^{-rT}Min\{S_T, qe^{\gamma T}\}I_{[\tau>T]} + e^{-r\tau}C_1 e^{-\delta(T-\tau)}I_{[\tau\leq T]}] \quad (2.9)$$

where, τ is defined as in equation (2.8).

The Black and Cox specification of the parameter imply that $C_1 < qe^{\gamma T}$. For the purpose of our model, we can take $C_1 = qe^{\gamma T}$, without any loss of generality. Nevertheless, we note that,

$$q > C_1 e^{-\delta T}, \text{ since } \delta > \gamma. \quad (2.10)$$

Let us denote $(qe^{\gamma T} - S_T)^+ = Max\{(qe^{\gamma T} - S_T), 0\}$. Then, the expression $Min\{S_T, qe^{\gamma T}\}$ can be decomposed and rewritten as:

$$Min\{S_T, qe^{\gamma T}\} = qe^{\gamma T} - (qe^{\gamma T} - S_T)^+$$

Using the above decomposition, the expression for the valuation of the contract at $t = 0$ can be rewritten as:

$$V_0 = E[e^{-rT}\{qe^{\gamma T} - (qe^{\gamma T} - S_T)^+\}I_{[\tau>T]} + e^{-r\tau}C_1 e^{-\delta(T-\tau)}I_{[\tau\leq T]}]$$

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[4]From the theory of option pricing, it is well known that in an economy consisting of two assets, the price C_0 at time zero of any contingent claim paying $C(S_T)$ at time T is equal to:

$$C_0 = E^Q[e^{-rT}C(S_T)].$$

where, $\mathbb{Q}$ is the EMM under which the stock price follows a GBM. [For further details, please refer to Lando, D: "Credit Risk Modeling", PP 9]

which, essentially boils down to:

$$V_0 = qe^{-(r-\gamma)T}E[I_{[\tau>T]}] - E[e^{-rT}(qe^{\gamma T} - S_T)^+ I_{[\tau>T]}] + C_1 e^{-\delta T}E[e^{-(r-\delta)\tau}I_{[\tau\leq T]}] \qquad (2.11)$$

It is evident from the above expression that the probability distribution of the *first passage time* τ plays a crucial role in computing the valuation. To this end, we first state and prove the central result of this section that calculates the probability distribution function of the stopping time . This distribution essentially is the probability distribution of first passage time of a Brownian motion with drift, which can be derived by collecting facts and formulas about Brownian motion.[5] For the purpose of completeness, a proof is included.

Lemma 1. *The probability distribution of the first passage time τ, defined as in equation (2.8) is given by:*

$$\mathbb{Q}[\tau \leq T] = N(\frac{a - \mu T}{\sigma\sqrt{T}}) + e^{2\mu a\sigma^{-2}}N(\frac{a + \mu T}{\sigma\sqrt{T}}) \qquad (2.12)$$

where, $\mu = r - \delta - \dfrac{\sigma^2}{2}; \ a = \ln(\dfrac{C_1}{S_0}) - \delta T, \ and \ N(x) = \dfrac{1}{2\pi}\int\limits_{-\infty}^{x} e^{-\frac{1}{2}z^2}dz.$

Proof. The default time τ, as introduced in the equation (2.8), is given by,

$$\tau = \inf\{t \geq 0 : S_t \leq C_1 e^{-\delta(T-t)}\}$$

Using a simple algebraic manipulation, this stopping time can be expressed in terms of a transformed process X_t as:

$$\tau = \inf\{t \geq 0 : X_t \leq C_1 e^{-\delta T} = C_T\} \qquad (2.13)$$

where, $X_t = e^{-\delta t}S_t$, follows the SDE:

$$dX_t = X_t[(r - \delta)dt + \sigma dW_t] \qquad (2.14)$$

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[5]Please refer to Borodin and Salminen [3] for useful "Facts and Formulas about Brownian motion". Detail derivations of similar formulas is presented in Bielecki and Ruthkowski [2].

with $X_0 = x$. Using Ito's lemma, we can obtain that the X_t process follows:

$$X_t = x \exp\{(r - \delta - \frac{\sigma^2}{2})t + \sigma W_t\} \tag{2.15}$$

Next, a logarithmic transformation is used to represent the stopping time τ in terms of a passage time of a Brownian motion with drift, as:

$$\begin{aligned} \tau &= \inf\{t \geq 0 : \ln V_t \leq \ln C_T\} \\ &= \inf\{t \geq 0 : [(r - \delta - \frac{\sigma^2}{2})t + \sigma W_t] \leq \ln(\frac{C_T}{V_0})\}. \end{aligned}$$

Denoting $(r - \delta - \frac{\sigma^2}{2}) = \mu$, and $\ln(\frac{C_1}{S_0}) - \delta T = a$ we get:

$$\tau = \inf\{t \geq 0 : \mu t + \sigma W_t \leq a\}.$$

In words, τ is the first time when this Brownian motion with drift becomes less than or equal to the constant value a. Now, if that random time happens to be greater than T, then simple mathematical logic implies that, the minimum value that the process attains within the entire interval $[0, T]$ must be greater than a. Translating this logic into equations, we get:

$$\mathbb{Q}[\tau > T] = \mathbb{Q}[\inf_{0 \leq t \leq T} (\mu t + \sigma W_t) > a] \tag{2.16}$$

We invoke the distribution of the running infimum process of a Brownian motion with drift to get the desired probability distribution. For $\sigma > 0$, $\mu \in \mathbb{R}$ and any $a > 0$, the distribution of the running infimum process is given by:

$$\mathbb{Q}[\inf_{0 \leq t \leq T} (\mu t + \sigma W_t) > a] = N(\frac{-a + \mu T}{\sigma\sqrt{T}}) - e^{2\mu a \sigma^{-2}} N(\frac{a + \mu T}{\sigma\sqrt{T}}) \tag{2.17}$$

where, $N(u) = \frac{1}{2\pi} \int_{-\infty}^{u} e^{-\frac{1}{2}z^2} dz$ is a standard normal distribution function (Borodin and Salminen, 2002). Henceforth, the remaining part of the deduction is straightforward. The *probability distribution function of the first passage time τ with respect to the risk neutral probability is*

given by:

$$\mathbb{Q}[\tau \leq T] \quad = \quad 1 - \mathbb{Q}[\inf_{0 \leq t \leq T}(\mu t + \sigma W_t) > a] \tag{2.18}$$

$$= \quad N\left(\frac{a - \mu T}{\sigma\sqrt{T}}\right) + e^{2\mu a \sigma^{-2}} N\left(\frac{a + \mu T}{\sigma\sqrt{T}}\right) \tag{2.19}$$

since, $1 - N\left(\frac{-a + \mu T}{\sigma\sqrt{T}}\right) = N\left(\frac{a - \mu T}{\sigma\sqrt{T}}\right)$, follows from the symmetry. This completes the proof.

$\square$

Thus, using the *probability distribution of the first passage time of a Brownian motion with drift*, we can easily calculate an analytical expression for the first term of equation (2.11):

$$qe^{-(r-\gamma)T} E[I_{[\tau > T]}] = qe^{-(r-\gamma)}\left[N\left(\frac{-a + \mu T}{\sigma\sqrt{T}}\right) - e^{2\mu a \sigma^{-2}} N\left(\frac{a + \mu T}{\sigma\sqrt{T}}\right)\right] \tag{2.20}$$

Moreover, the following term the the equation (2.11), $C_1 e^{-\delta T} E[e^{-(r-\delta)\tau} I_{[\tau \leq T]}]$ is simply an expectation of a function of a random variable, τ, whose distribution has already been derived. So, calculating the expectation is straightforward, as described below:

$$C_1 e^{-\delta T} E[e^{-(r-\delta)\tau} I_{[\tau \leq T]}] \quad = \quad C_1 e^{-\delta T} \int_{-\infty}^{T} e^{-(r-\delta)\tau} d\mathbb{Q}[\tau \leq T] \tag{2.21}$$

$$= \quad C_1 e^{-\delta T} \int_{-\infty}^{T} e^{-(r-\delta)\tau} d\left[N\left(\frac{a - \mu T}{\sigma\sqrt{T}}\right) + e^{2\mu a \sigma^{-2}} N\left(\frac{a + \mu T}{\sigma\sqrt{T}}\right)\right] \tag{2.22}$$

The remaining part of the valuation (i.e., the last term in the RHS of the equation (2.11)) $E[e^{-rT}(qe^{\gamma T} - S_T)^+ I_{[\tau > T]}]$, is exactly same as the valuation of a rigorously studied and quite popular financial derivative, called *down-and-out barrier option*. Since the analytical solution for the valuation of *down-and-out barrier option* already exists in the literature, we simply

borrow the term for our purpose, which takes the following form:[6]

$$E[e^{-rT}(qe^{\gamma T} - S_T)^+ I_{[\tau > T]}] = e^{-(r-\delta)T} \int_{\ln C_T}^{\infty} (q_T - e^p)\varphi(p; \mu T + \ln(S_0), \sigma\sqrt{T})dp \qquad (2.23)$$

$$+ e^{-(r-\delta)T}(\frac{C_T}{S_0})^{2\mu\sigma^{-2}} \int_{\ln C_T}^{\infty} (q_T - e^p)\varphi(p; \mu T + \ln(\frac{C_T^2}{S_0}), \sigma\sqrt{T})dp \qquad (2.24)$$

where, where, $\mu = r - \delta - \frac{\sigma^2}{2}$ and $\varphi(x; \mu, \sigma)$ denotes the density of a normal distribution with mean μ and variance σ^2. Thus, introducing the risk of bankruptcy in the stock loan problem following Black and Cox (1976) formulation, we can find analytical solution for the valuation explicitly in terms of *probability distribution of the first passage time of Brownian motion with drift* and *the valuation of down-and-out barrier options*. The closed form expression for the valuation is:

$$V_t = qe^{-(r-\gamma)}[N(\frac{-a + \mu T}{\sigma\sqrt{T}}) - e^{2\mu a\sigma^{-2}}N(\frac{a + \mu T}{\sigma\sqrt{T}})]$$

$$- e^{-(r-\delta)T} \int_{\ln C_T}^{\infty} (q_T - e^p)\varphi(p; \mu T + \ln(S_0), \sigma\sqrt{T})dp$$

$$- e^{-(r-\delta)T}(\frac{C_T}{S_0})^{2\mu\sigma^{-2}} \int_{\ln C_T}^{\infty} (q_T - e^p)\varphi(p; \mu T + \ln(\frac{C_T^2}{S_0}), \sigma\sqrt{T})dp$$

$$+ C_1 e^{-\delta T} \int_{-\infty}^{T} e^{-(r-\delta)\tau}d[N(\frac{a - \mu T}{\sigma\sqrt{T}}) + e^{2\mu a\sigma^{-2}}N(\frac{a + \mu T}{\sigma\sqrt{T}})]$$

In the *structural form modeling*, agent has complete information about the history of asset prices as well as the value of default barrier. Given that information, agent can construct a sequence of stopping times (τ_n) in such a way that the sequence (τ_n) increases to the optimal stopping time τ^* as n approaches infinity. In other words, bankruptcy is modeled as predictable stopping time, given the knowledge of history of stock price. This specific characteristic faces severe criticism when compared to more sophisticated models of bankruptcy that models default as non predictable. Modeling bankruptcy as a non predictable event was first introduced in

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[6]An extensive coverage on barrier option can be found in Bjork (2004). For further reference, please refer to Rubinstein and Reiner (1991) and the survey in Carr (1995).

reduced form modeling (Jarrow and Turnbull, 1995), which is also termed as intensity modeling interchangeably.

2.4   Bankruptcy: Intensity Modeling

In the general *reduced-form or intensity modeling approach*, we deal with two kinds of information: the information conveyed by the assets prices, and the information about the occurrence of the default time. The later refers to the knowledge of the time where the default occurred in the past, if the default has indeed already happened. The later is a bigger filtration which is generated by the default process and incorporates the information conveyed by the observed asset price process. This specific feature makes the default non-predictable. In this formulation, finding the value of the contract at time zero is a much more challenging mathematical problem.

The fundamental idea of the intensity based framework is to model default occurring according to an intensity function. In the literature, the intensity function is modeled as a decreasing function of the stock price, which essentially represents the instantaneous default probability, i.e. the (very) short-term default risk. The notion comes form the operation research literature where, the intensity function or hazard rate is the conditional default arrival rate, given no default so far.

$$\lim_{\delta \to 0} \frac{\mathbb{P}[\tau_0 \in (t, t+\delta)|\tau_0 > t]}{\delta} = h(S_t) \tag{2.25}$$

The argument behind the construction of the intensity function is very intuitive. Higher is the current stock price, less is the risk of default which is in conformity with the value of the intensity function. Now this intuition holds true as long as the inverse relationship between current stock price and default intensity is maintained. To facilitate our analysis, we make some assumptions regarding the intensity function and its properties.

2.4.1 Assumptions

(i) In order to introduce the hazard rate intensity function $(h(.))$,we assume that h is a continuously differentiable, strictly decreasing function defined on $(0, \infty)$. The function exhibits following characteristics:

$$\lim_{x \to 0^+} h(x) = +\infty \text{ and } \lim_{x \to +\infty} h(x) = 0. \tag{2.26}$$

(ii) We assume that the hazard rate intensity function is a function of the discounted stock price, where the discount factor is the lending interest rate γ. We denote the discounted stock price $e^{-\gamma t} S_t$ by $\tilde{S}_t$, which represents the value of the stock to the lender. We assume that the lender (the Bank) decides the worth of the stock at each point in time while modeling bankruptcy and therefore, consider the discounted value of stock price.

$$h(e^{-\gamma t} S_t) = h(\tilde{S}_t). \tag{2.27}$$

Defining the intensity function in this manner is in conformity with the spirit of intensity modeling. To elaborate, as the stock price declines towards zero, so does the discounted stock price and as a result, default intensity blows up to infinity. As the stock price go up, so does the discounted stock price and consequently, hazard rate declines to zero, making the stock price process asymptotically tend to a GBM.

2.4.2 Fundamentals

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a risk neutral probability space supporting a standard Brownian motion $\{W_t, t \geq 0\}$. The information revealed by the Brownian motion at every instant t is captured by the Brownian filtration $(\mathcal{F}_t)_{t \geq 0}$, with $\mathcal{F}_0 = \sigma(\phi, \Omega)$ and $\mathcal{F} = \sigma(\cup_{t \geq 0} \mathcal{F}_t)$. We assume a frictionless market, no arbitrage and take an equivalent martingale measure (EMM) given as $\mathbb{Q}$. To model bankruptcy as an inaccessible phenomenon, an exponential random variable e with parameter 1 is introduced that is independent to the Brownian motion. Random time

of bankruptcy τ_0 is modeled as the first time when the process $\int_0^t h(\tilde{S}_u)du$ is greater or equal to the realization of independent random variable e. In words, the stock is considered as bankrupt when the accumulated conditional probability of default exceeds the realization of an exponential random variable, independent to the Brownian motion.

$$\tau_0 = \inf\{t \geq 0 : \int_0^t h(\tilde{S}_u)du \geq e\} \tag{2.28}$$

$$= +\infty, \text{ if the above set is empty.} \tag{2.29}$$

At the time of bankruptcy τ_0, the stock price jumps to the bankruptcy state $\triangle$, where it remains forever. In the terminology of the Markov processes, $\triangle$ is called a *cemetery state*. We assume that, the stock owner does not get any recovery in the event of bankruptcy and the stock becomes worthless. Therefore, we model the stock price subject to bankruptcy as a diffusion process $\{S_t^{\triangle}, t \geq 0\}$ with the extended state space $E^{\triangle} = (0, \infty) \cup \{\triangle\}$, diffusion coefficient σS_t, drift $(r + h(\tilde{S}_t))S_t$ and hazard rate $h(\tilde{S}_t)$.[7] To ensure that the discounted stock price is a martingale under EMM $\mathbb{Q}$, the hazard rate intensity needs to be added to the drift of the process (Davis and Lischka, 2002). Intuitively, addition of the hazard rate to the drift in the pre-default stock price process increases the expected rate of return from the defaultable stock, to provide the stockholder with some incentive to compensate for additional risk. In our notation, $\{S_t, t \geq 0\}$ is pre-default stock price process, whereas, $\{S_t^{\triangle}, t \geq 0\}$ is the stock price process subject to bankruptcy. Therefore,

$$S_t^{\triangle} = S_t, \text{ for } t < \tau_0$$

$$= \triangle, \text{ for } t \geq \tau_0.$$

To describe the dynamics of the asset price process $\{S_t^{\triangle}, t \geq 0\}$ and to keep track of how information is revealed over time, we introduce the bankruptcy jump indicator process $\{D_t :$

---

[7]For details, please refer to Elliot, Jeanblanc and Yor (2001) and Linetsky (2004).

$t \geq 0\}$ by

$$D_t = I_{[\tau_0 \leq t]}$$

Let $\{\mathcal{D}_t, t \geq 0\}$ be the filtration generated by the (D_t) process, and $\{\mathcal{F}_t, t \geq 0\}$ be the filtration generated by the brownian motion process. We introduce an enlarged filtration, $\mathcal{G}_t = \mathcal{F}_t \cup \mathcal{D}_t$, for each $t \geq 0$. The defaultable stock price is adapted to $\mathcal{G}_t$ and therefore, τ_0 is a $\mathcal{G}_t$ -stopping time. It is customary to identify the *cemetery state* $\triangle = 0$. Then we can represent the defaultable stock price process $\{S_t^{\triangle}, t \geq 0\}$ by,

$$dS_t^{\triangle} = S_{t-}^{\triangle}[rdt + \sigma dW_t - dM_t]$$

where,

$$M_t = D_t - \int\limits_0^{\min\{\tau_0, t\}} h(\tilde{S}_u)du$$

We note that, the process $\{M_t : t \geq 0\}$ is a $(\mathcal{G}_t)$ martingale. Therefore, $e^{-rt}S_t^{\triangle}$ is also a $(\mathcal{G}_t)$ martingale.

The pre-default underlying asset price dynamics under the EMM follows a diffusion process, satisfying the following stochastic differential equation (SDE):

$$dS_t = S_t[(r + h(\tilde{S}_t))dt + \sigma dW_t] \tag{2.30}$$

with $S_0 = x$, where, $h(\tilde{S}_t)$ denotes the state dependent default intensity. The SDE that the pre-default stock price follows has a unique strong non-exploding solution under the assumption (i).

2.4.3   Problem Formulation

The client owns a share of a stock with the price process at time t is given by $S_t^{\triangle}$, which is subject to bankruptcy and is modeled by above SDE's. The information available to the investor at time t is represented by $\sigma-$algebra $\mathcal{F}_t$, which makes τ_0 non-predictable. The rate

of time preference is assumed to be $\rho > r$. Without any loss of generality, ρ is assumed to equal $(r + \epsilon)$, where ϵ is a small positive number. Therefore, the problem of the stock owner is to find a $(\mathcal{F}_t)$−stopping time τ^* so as to maximize $E[e^{-\rho\tau}(S_\tau^\triangle - qe^{\gamma\tau})^+ I_{[\tau<\tau_0]}]$ over all the $(\mathcal{F}_t)$−stopping time τ. Since the information available to the owner at time t is represented by $\mathcal{F}_t$, we maximize the expected discounted payoff over all $(\mathcal{F}_t)$-stopping times τ. The value function is given by:

$$V(x) = \sup_\tau E_x[e^{-\rho\tau}(S_\tau^\triangle - qe^{\gamma\tau})^+ I_{[\tau<\tau_0]}]. \tag{2.31}$$

where, the supremum is taken over all the $(\mathcal{F}_t)$−stopping times τ. The following mathematical result is used to reduce the problem in terms of the pre-default stock price. This result is the same as Proposition 3.1 in Meng and Weerasinghe (2007). We present an alternative proof here for the purpose of completeness.

Proposition 2. *For any finite $\mathcal{F}_t$ stopping time τ,*

$$E[I_{[\tau<\tau_0]}|\mathcal{F}_\tau] = e^{-\int_0^\tau h(\tilde{S}_u)du} \tag{2.32}$$

where, τ_0 is the bankruptcy time, as introduced in (2.29).

Proof. The bankruptcy time τ_0 is introduced as:

$$\tau_0 = \inf\{t \geq 0 : \int_0^t h(\tilde{S}_u)du \geq e\}$$

Using this definition, we have,

$$
\begin{aligned}
E[I_{[\tau<\tau_0]}|\mathcal{F}_\tau] &= \mathbb{Q}[(\tau < \tau_0)|\mathcal{F}_\tau] \\
&= \mathbb{Q}[\int_0^\tau h(\tilde{S}_u)du \leq e|\mathcal{F}_\tau] \\
&= e^{-\int_0^\tau h(\tilde{S}_u)du}
\end{aligned}
$$

The last equality follows from the fact that $\int_0^\tau h(\tilde{S}_u)du$ is a known constant given the information

revealed by $\mathcal{F}_\tau$, and e is an exponential random variable with mean 1 independent of $\mathcal{F}_\tau$. Hence the result. $\qquad\square$

Using this proposition, we can reduce the problem in terms of the pre-default stock price, as:

$$
\begin{aligned}
E[e^{-\rho\tau}(S_\tau^\triangle - qe^{\gamma\tau})^+ I_{[\tau<\tau_0]}] &= E[e^{-\rho\tau}(S_\tau - qe^{\gamma\tau})^+ I_{[\tau<\tau_0]}] \\
&= E[e^{-\rho\tau}(S_\tau - qe^{\gamma\tau})^+ I_{[\tau<\tau_0]}|\mathcal{F}_\tau] \\
&= E[e^{-\int_0^\tau(\rho+h(\tilde{S}_u))du}(S_\tau - qe^{\gamma\tau})^+]
\end{aligned}
$$

Consequently, the value function can be represented as :

$$
V(x) = \sup_\tau E[e^{-\int_0^\tau(\rho+h(\tilde{S}_u))du}(S_\tau - qe^{\gamma\tau})^+]
$$

where, the supremum is taken over all the $(\mathcal{F}_t)-$stopping times τ. Using the transformation $e^{-\gamma t}S_t = \tilde{S}_t$, we can express the value function as a function of $\tilde{S}_t$.

$$
V(x) = \sup_\tau E[e^{-\int_0^\tau(\rho-\gamma+h(\tilde{S}_u))du}(\tilde{S}_\tau - q)^+] \tag{2.33}
$$

Thus, without any loss of generality, the optimal stopping problem reduces to find a $(\mathcal{F}_t)$-stopping time τ so as to maximize the value function:

$$
V_x = \sup_\tau E[e^{-\int_0^\tau(\tilde{r}+\epsilon+h(\tilde{S}_u))du}(\tilde{S}_\tau - q)^+] \tag{2.34}
$$

where, $\tilde{r} = (r - \gamma) < 0$. The transformed stock price $\tilde{S}_t$ satisfies the following SDE:

$$
d\tilde{S}_t = \tilde{S}_t[(\tilde{r} + h(\tilde{S}_t))dt + \sigma dW_t] \tag{2.35}
$$

$$
\tilde{S}_0 = x \tag{2.36}
$$

At this point, the counteractive forces governing the dynamics behind the optimization

problem is prominent. As the stock price fluctuates, it spills the impact over the intensity function which in turn initiate a perturbation in the model from two avenues. A sharp decline in the stock price raises the intensity value and thus increases the expected rate of return form the stock. This effect provides an incentive for the owner the hold the stock a bit longer. On the flip side, increase in intensity makes the future gain getting discounted more heavily and acts as a disincentive to hold the option. Since the absolute value of the discount factor exceeds that of the drift term, the disincentive generated from a stock price decline tends to dominate in effect. Nonetheless, this underlying tradeoff makes the economic decision non trivial and the problem interesting to analyze.

2.4.4 A verification lemma

Following the method of dynamic programming for the optimal stopping problem, we can write down the formal Hamilton-Jacobi-Bellman (HJB) equation associated with our problem as:

$$\max\{\frac{\sigma^2}{2}x^2 Q''(x) + (\tilde{r} + h(x))xQ'(x) - (\tilde{r} + \epsilon + h(x))Q(x), (x - q)^+ - Q(x)\} = 0 \qquad (2.37)$$

for almost all $x \in [0, \infty)$. Hence, the derivation of an optimal stopping time is closely associated with finding a smooth solution to HJB equation (2.37). Our objective is to show that value function $V(x)$ is the unique smooth solution to the HJB equation above. In order to show that, we first state and proof the following verification lemma:

Lemma 3 (Verification Lemma). *Let Q be a non-negative, continuously differentiable function defined on $[0, \infty)$, which is piecewise twice differentiable. Also, the limits $\lim_{c^-} Q''(x)$ and $\lim_{c^+} Q''(x)$ exists and are finite for every c. Assume that the function Q satisfies the HJB equation above for almost all $x \in [0, \infty)$. Then $Q(x) \geq V(x)$ for all $x > 0$, where $V(.)$ is the value function described by:*

$$V(x) = \sup_{\tau} E[e^{-\int_0^\tau (\tilde{r} + \epsilon + h(\tilde{S}_u))du}(\tilde{S}_\tau - q)^+].$$

Proof. We define the the differential operator $\mathcal{L}$ by:

$$\mathcal{L} = \frac{\sigma^2}{2}x^2\frac{d^2}{dx^2} + (\tilde{r} + h(x))x\frac{d}{dx} - (\tilde{r} + \epsilon + h(x)) \tag{2.38}$$

for all $x > 0$. Let $S_0 = x$ is fixed. For each $n > x$, we introduce a sequence (τ_n) of $(\mathcal{F}_t)-$ stopping times by:

$$\begin{aligned}
\tau_n &= \inf\{t \geq 0 : \tilde{S}_t \geq n\} \\
&= +\infty, \text{ if the above set is empty.}
\end{aligned}$$

Then as $n \to \infty$, the sequence of stopping times (τ_n) also increases to infinity. Using Ito's lemma, we obtain:

$$E_x[e^{-\int_0^{\tau\wedge\tau_n}(\tilde{r}+\epsilon+h(\tilde{S}_u))du}Q(S_{\tau\wedge\tau_n})] = Q(x) + E_x\int_0^{\tau\wedge\tau_n}e^{-\int_0^u(\tilde{r}+\epsilon+h(\tilde{S}_r))dr}(\mathcal{L}Q)(\tilde{S}_u)du$$

The last inequality holds since $\mathcal{L}Q \leq 0$ by the HJB equation. Since Q is a non-negative function, applying Fatou's lemma gives us

$$E_x[e^{-\int_0^{\tau}(\tilde{r}+\epsilon+h(\tilde{S}_u))du}Q(\tilde{S}_\tau)] \leq Q(x), \text{ for each } x \geq 0.$$

Next, we can take the supremum over all $(\mathcal{F}_t)-$stopping times τ and obtain $V(x) \leq Q(x)$, for each $x \geq 0$. This completes the proof. $\qquad\square$

In the next section, the existence of the optimal exercise boundary of this financial derivative is proved. Moreover, it is shown that the optimal exercise boundary is of threshold type. In other words, it is optimal to regain the stock at the first time the stock price reached a closed interval $[x^*, \infty)$. To prove our central theorem, first the existence of a smooth solution of (2.37) and the existence of a free boundary point is assumed. This solution and the boundary point is used to construct a optimal selling time. After this, what remains to prove is the existence

of such a function and the optimal exercise boundary. This is proved using a pamametrization method and considering a transformed problem.

2.4.5   Optimal Exercise Boundary

Let us assume the existence of a point x^* and a non negative smooth function $\hat{Q}$ such that the function is defined on the $(0, \infty)$ and it satisfies:

$$\mathcal{L}\hat{Q} = 0 \text{ for all } x > 0, \ \hat{Q}(x^*) = (x^* - q)^+, \ \hat{Q}'(x^*) = 1$$

and $\hat{Q}''(x^*) > 0$. Moreover, $\hat{Q}(x) > max\{x - q, 0\}$ for all $0 < x < x^*$ and $\hat{Q}(0+) = 0$. Then we introduce the optimal exercise function as:

$$Q^*(x) \ = \ \hat{Q}(x) \quad \text{for } 0 < x < x^* \tag{2.39}$$

and,

$$Q^*(x) = \ (x^* - q) \quad \text{for } x \geq x^* \tag{2.40}$$

Next, we state and prove the central result of the section regarding the value function and the optimal exercise boundary. The issue of existence of such a function and the free boundary problem will be addressed in the next section.

Theorem 4. *Let Q^* be defined as in (2.39) and (2.40). Then*

  (i) Q^* is a continuously differentiable function on $[0, \infty)$ which satisfies the HJB equation, given in (2.37). Moreover, $Q^*(.)$ is twice continuously differentiable everywhere except at the optimal exercise point x^* and $Q^{*''}(x^*+)$ and $Q^{*''}(x^*-)$ are finite.

  (ii) Let τ^* be a $(\mathcal{F}_t)$ stopping time defined by:

$$\tau^* \ = \ \inf\{t > 0 : \tilde{S}_t \geq x^*\} \tag{2.41}$$

$$= \ +\infty, \text{ if the above set is empty.} \tag{2.42}$$

Then τ^* is an optimal stopping time and,

$$V(x) = Q^*(x) \quad for\ all\ x \geq 0.$$

where $V(x)$ is the value function.

Proof. By construction, Q^* coincides with the function $\hat{Q}$ which satisfies HJB equation (2.37) on the interval $(0, x^*)$. Clearly, Q^* satisfies the HJB equation (2.37) on $(0, x^*)$. To verify that Q^* satisfies the HJB equation on the interval (x^*, ∞), we notice that $\mathcal{L}Q^*(x^*-) = \mathcal{L}\hat{Q}(x^*-) = 0$ and hence, $\hat{Q}''(x^*-) > 0$. Therefore,

$$\epsilon x^* - q(\tilde{r} + \epsilon + h(x^*)) > 0$$

Consequently,

$$\epsilon x - q(\tilde{r} + \epsilon + h(x)) > \epsilon x^* - q(\tilde{r} + \epsilon + h(x^*)) > 0, \quad for\ all\ x > x^*$$

since, $h'(.) < 0$. Therefore, since $Q^*(x) = x - q$ for all $x > x^*$, we have,

$$\mathcal{L}Q^* < 0 \quad \text{on the interval } (x^*, \infty).$$

This completes the proof of part (i).

Proof of part (i) along with Lemma (3) implies that

$$Q^*(x) \geq V(x) \quad for\ all\ x > 0.$$

To complete the proof of part (ii), it remains to show that

$$Q^* \leq V(x) \quad for\ all\ x > 0.$$

We first consider the case $x < x^*$. Let τ^* be as in (2.42). Then using Ito's lemma on the

function $Q^*(\tilde{S}_t)e^{-\int_0^t(\tilde{r}+\epsilon+h(\tilde{S}_u)du}$, we obtain,

$$E[Q^*(\tilde{S}_{T\wedge\tau^*})e^{-\int_0^{T\wedge\tau^*}(\tilde{r}+\epsilon+h(\tilde{S}_u))du}] = Q^* + E\int_0^{T\wedge\tau^*}e^{-\int_0^t(\tilde{r}+\epsilon+h(\tilde{S}_u))du}\mathcal{L}Q^*(\tilde{S}_t)dt$$

Since Q^* is bounded on $[0, x^*]$, $\mathcal{L}Q^*(x) = 0$ when $x < x^*$ and $Q^*(x^*) = x^* - q$, then letting T tend to ∞, we obtain,

$$Q^*(x) = E[Q^*(\tilde{S}_{\tau^*})e^{-\int_0^{\tau^*}(\tilde{r}+\epsilon+h(\tilde{S}_u)du}] = E[e^{-\int_0^{\tau^*}(\tilde{r}+\epsilon+h(\tilde{S}_u)du}(\tilde{S}_{\tau^*} - q)^+]$$

when $x < x^*$. Since τ^* is a $(\mathcal{F}_t)$ stopping time, it follows that $Q^*(x) \leq V(x)$ when $x < x^*$. On the other hand, when $x \geq x^*$, $Q^*(x) = x - q$ and by definition, $\tau^* = 0$. Thus, $Q^*(x) \leq V(x)$ for $x \geq x^*$.

Hence, $Q^*(x) = V(x)$ for all x and τ^* is an *optimal stopping time*. Hence the theorem. □

2.4.6   Existence of the Function $\hat{Q}(.)$

To complete the proof of the central theorem, Theorem (4), it remains to prove the existence of such a function $\hat{Q}$ and an associated optimal stopping boundary point x^*. To this end, a parametric family of functions that satisfy the second order linear homogeneous ordinary differential equation $\mathcal{L}Q = 0$ will be analyzed. Finally, the optimal function will be chosen using the solution to a *variational equality and principle of smooth-fit*.

First, we consider a logarithmic transformation $Y_t = \ln(\tilde{S}_t)$, with the transformed initial value $y = \ln(x)$, primarily because of analytical convenience. The discounted stock price $\tilde{S}_t$ satisfies the SDE specified in (2.36). Using Ito's lemma, it is straightforward to find that Y_t process satisfies:

$$dY_t = (\tilde{r} - \frac{\sigma^2}{2} + \phi(Y_t))dt + \sigma dW_t \tag{2.43}$$

with $Y_0 = y$, where $\phi(y)$ represents the transformed default intensity $h(e^y)$, which is defined on the domain $(-\infty, +\infty)$. The assumptions imposed on the characteristics of $h(.)$ function implies that $\lim_{y\to-\infty}\phi(y) = +\infty$ and $\lim_{y\to\infty}\phi(y) = 0$. The differential operator $\mathcal{H}$ associated with

the transformed process Y_t is given by:

$$\mathcal{H} = \frac{\sigma^2}{2}\frac{d}{d^2 y} + (\tilde{r} - \frac{\sigma^2}{2} + \phi(y))\frac{d}{dy} - (\tilde{r} + \epsilon + \phi(y)) \tag{2.44}$$

Using the logarithmic transformation, the initial problem of finding a function $\hat{Q}$ on the domain $(0, \infty)$ and an associated optimal exercise boundary x^* is transformed to finding a function U and an associated point y^*, such that:

(i) The function $U(.)$ is positive, strictly increasing, continuously differentiable and $\lim\limits_{y \to -\infty} U(y) = 0$.

(ii) Moreover, $\mathcal{H}U(y) = 0$ for all $y < y^*$, $U(y) = (e^y - q)^+$ for $y \geq y^*$, $U'(y^*) = e^{y^*}$ and $U''(y^*-)$ is finite.

Then, the relationship between the original and the transformed value functions can be expressed as $\hat{Q}(x) = U(\ln(x))$.

To prove the existence of such a function in our analysis, we introduce a point $\hat{y}$ by:

$$\hat{y} = \inf\{y > 0 : \tilde{r} + \phi(y) \leq 0\} \tag{2.45}$$

At this point, we make a minor assumption about the structure of our environment. We assume that the $\hat{y}$ defined in equation (2.45) satisfies $\hat{y} > \ln q$. This is a minor assumption and tends to be satisfied in most of the environments without any major modification. We note here that existence results are not crucially contingent on this assumption. Assuming this, we would show that for any large b, such that $\hat{y} > b > \ln(q)$, the boundary value problem

$$\mathcal{H}U_b(y) = 0, \ U_b(b) = (e^b - q) \ and \ \lim\limits_{y \to -\infty} U_b(y) = 0 \tag{2.46}$$

has a unique solution $U_b(y)$. Moreover, this unique solution has the following stochastic representation, which can be derived using Ito's lemma (as will be shown subsequently):

$$U_b(y) = E_y[e^{-\int_0^{\tau_b}(\tilde{r}+\epsilon+\phi(y_u))du}](e^b - q)^+ \tag{2.47}$$

To this end, for each b within the interval $\hat{y} > b > \ln(q)$, an $(\mathcal{F}_t)$ stopping time τ_b is introduced as:

$$\tau_b = \inf\{t \geq 0 : Y_t \geq b\} \tag{2.48}$$

$$= +\infty, \text{ if the above set is empty} \tag{2.49}$$

Also, a function $U_b(y)$ defined on the interval $(-\infty, b]$ is introduced for every $b \in [\ln(q), \hat{y}]$ by:

$$U_b(y) = E_y[e^{-\int_0^{\tau_b}(\tilde{r}+\epsilon+\phi(y_u))du}](e^b - q) \tag{2.50}$$

Since the function $U_b(.)$ is positive and increasing throughout the domain (as will be shown subsequently), it intersects the curve $(e^y - q)^+$ at a point b, where $b > \ln(q)$. Moreover, since we consider any b where $\hat{y} > b > \ln(q)$, $(e^y - q)^+$ is equivalent to the amount $(e^y - q)$.

The next lemma shows that the function $U_b(y)$ is the unique solution to the boundary value problem specified in (2.46).

Lemma 5. *Let $\hat{y}$ be defined as in the equation (2.45) and also, let us assume $\hat{y} > \ln(q)$. The (Y_t) process follows the SDE (2.43). For any b, such that $\hat{y} > b > \ln(q)$, and for each $n > |y|$, the stopping time τ_n is introduced by:*

$$\tau_n = \inf\{t \geq 0 : Y_t \leq -n \text{ or } Y_t \geq b\} \tag{2.51}$$

$$= +\infty, \text{ if the above set is empty} \tag{2.52}$$

For every $y \leq b$, it is assumed that :

$$U_n(y) = E_y[e^{-\int_0^{\tau_n}(\tilde{r}+\epsilon+\phi(y_u))du}I_{[Y_{\tau_n}=b]}](e^b - q) \tag{2.53}$$

Then the following results hold:

(i) *$U_n(y)$ is the unique solution to the boundary value problem $\mathcal{H}U_n(y) = 0$ for all $-n < y < b$, $U_b(b) = e^b - q$ and $U_n(-n) = 0$.*

(ii) U_n has no local extrema and $U_n' > 0$ on the interval $[-n, b)$.

(iii) For any fixed $y < b$, the sequence $(U_n(y))$ is strictly increasing in n.

(iv) $\lim\limits_{n \to \infty} U_n(y) = U_b(y)$ for each $y \le b$ where, $U_b(y)$ is given by the equation 2.47.

Proof. From the theory of ordinary differential equations, it follows that there is a unique solution to the boundary value problem given in part (i) of the lemma on the interval $[-n, b]$. For the stochastic representation, let τ_n be defined as in (2.52). Applying Ito's lemma to the function $U_n(y)$, we obtain:

$$E[U_n(Y_{T \wedge \tau_n})e^{-\int_0^{T \wedge \tau_n}(\tilde{r} + \epsilon + \phi(y_u))du}] = U_n(y) + E[\int_0^{T \wedge \tau_n} e^{-\int_0^t(\tilde{r} + \epsilon + \phi(y_u))du}\mathcal{H}U_n(Y_t)dt]$$

Now, since $U_n(y)$ is bounded and $\mathcal{H}U_n(y) = 0$ on $[-n, b]$, we let T tend to infinity to obtain:

$$U_n(y) = E[U_n(Y_{\tau_n})e^{-\int_0^{\tau_n}(\tilde{r} + \epsilon + \phi(y_u))du}]$$

Since $U_n(-n) = 0$ and $U_n(b) = e^b - q$, it follows that:

$$U_n(y) = E_y[e^{-\int_0^{\tau_n}(\tilde{r} + \epsilon + \phi(y_u)du} I_{[Y_{\tau_n} = b]}](e^b - q)$$

This completes the proof of part (1).

For part (ii), we first note that the stochastic representation implies that $U_n(y) > 0$ on the interval $[-n, b]$. Now, if there exists a $\zeta \in (-n, b]$, such that $U_n'(\zeta) = 0$, then $\mathcal{H}U_n(\zeta) = 0$ implies that $\frac{\sigma^2}{2}U_n''(\zeta) = (\tilde{r} + \epsilon + \phi(y))U_n(\zeta) > 0$ and hence ζ is necessarily a strict local minimum. Now, since $U_n(-n) = 0 < U_b(b) = e^b - q$ and $U_n(y) > 0$ on the interval $[-n, b]$ and any critical point is a strict local minimum, it simply implies $U_n'(.) > 0$ on the entire interval $[-n, b]$. To prove part (iii) of the lemma, we simply note that $U_{n+1}(b) - U_n(b) = 0$, $U_{n+1}(-n) > U_n(-n) = 0$ and $\mathcal{H}(U_{n+1} - U_n)(y) = 0$ on $(-n, b]$. Therefore, using maximum principle for differential equations,[8] it follows that $U_{n+1}(y) > U_n(y)$ on the interval $(-n.b]$.

[8]Please refer to Protter and Weinberger (1967) for a detail discussion on the Maximum Principle.

To prove part (iv), we consider the scale function $S(.)$ of the diffusion process (Y_t), which is given by:

$$S(y) = \int\limits_0^y e^{-\frac{\sigma^2}{2}\int_0^z (\tilde{r}-\frac{\sigma^2}{2}+\phi(u))du}\, dz \tag{2.54}$$

Then,

$$P_y[Y_{\tau_n} = b] = \frac{S(y) - S(-n)}{S(b) - S(-n)} \tag{2.55}$$

Since $\lim_{y\to-\infty} \phi(y) = +\infty$, it follows that $\lim_{n\to\infty} S(-n) = -\infty$. Consequently, we obtain $\lim_{n\to\infty} P_y[Y_{\tau_n} = b] = 1$. Using the stochastic representation and the bounded convergence theorem, the result follows. □

Henceforth, $U_b(y)$ is defined in (2.47) and satisfies the condition (iv) of Lemma (5). The next proposition extends the domain of the function $U_b(y)$ to the interval $(-\infty, b]$ and derives some useful properties.

Proposition 6. *Let us consider $\hat{y} > b > \ln(q)$, and the function $U_b(y)$ is defined on the interval $(-\infty, b]$. Then, the following assertions hold:*

(i) $U_b(y)$ satisfies $\mathcal{H}U_b(y) = 0$ for all $y < b$.

(ii) $U_b(y) > 0$ and $U_b'(y) > 0$ for all $y < b$. Moreover, $U_b(y) > 0$ is bounded on $(-\infty, b]$.

(iii) $U_b(y) > 0$ is the unique solution to the boundary value problem $\mathcal{H}U_b(y) = 0$ for all $y < b$, with the boundary conditions $U_b(b) = e^b - q$ and $\lim_{y\to-\infty} U_b(y) = 0$.

Proof. From the previous lemma, we know that the sequence of functions $\{U_n(y)\}$ is increasing to $U_b(.)$ on the interval $(-\infty, b]$. Let us first fix a $y < b$. By integrating $\mathcal{H}U_b(y) = 0$ twice on the interval $[y, b]$ and by using integration by parts with the terms with first derivative, we

obtain:

$$\frac{\sigma^2}{2}U_n'(b)(b-y) = \frac{\sigma^2}{2}[U_n(b) - U_n(y)] - (\tilde{r} - \frac{\sigma^2}{2} + \phi(b))U_n(b)(b-y) \tag{2.56}$$

$$+ \int_y^b (\tilde{r} - \frac{\sigma^2}{2} + \phi(u))U_n(u)du + \int_y^b \int_u^b U_n(s)\phi'(s)dsdu \tag{2.57}$$

$$+ \int_y^b \int_u^b (\tilde{r} + \epsilon + \phi(s))dsdu \tag{2.58}$$

Now, by the previous lemma, $0 < U_n(y) < e^b - q$ for all $y < b$ and $\lim_{n\to\infty} U_n(y) = U_b(y)$ for each $y \leq b$. Therefore, using bounded convergence theorem and the equation (2.58), it follows that $\lim_{n\to\infty} U_n'(b) = \lambda$ exists and λ is finite. Furthermore, the identity we obtain by replacing $U_n'(b)$ by λ and $U_n(.)$ by $U_b(.)$ in the equation (2.58) remains valid. Hence, we have,

$$\frac{\sigma^2}{2}\lambda(b-y) = \frac{\sigma^2}{2}[U_n(b) - U_b(y)] - (\tilde{r} - \frac{\sigma^2}{2} + \phi(b))U_n(b)(b-y) \tag{2.59}$$

$$+ \int_y^b (\tilde{r} - \frac{\sigma^2}{2} + \phi(u))U_b(u)du + \int_y^b \int_u^b U_b(s)\phi'(s)dsdu \tag{2.60}$$

$$+ \int_y^b \int_u^b (\tilde{r} + \epsilon + \phi(s))dsdu \tag{2.61}$$

Next, we divide the equation by $(b-y)$ and let $y \to b$ and thus obtain $U_b'(b-) = \lambda$. By replacing replacing λ by $U_b'(b-)$ in the equation (2.61), and differentiating it twice, we obtain $\mathcal{H}U_b(y) = 0$. Hence, part (i) follows.

By parts (ii) and (iv) of Lemma (5), it follows that $U_b(y)$ is a non-decreasing on the interval $(-\infty, b]$, and hence, $U_b'(y) > 0$ for all $y < b$. Also, $U_b(y) > 0$ for all $y < b$ and $U_b(b) = e^b - q$. Suppose $U_b'(\zeta) = 0$ for some $\zeta < b$. Then, $\mathcal{H}U_b(\zeta) = 0$ together with the fact that $\zeta < b$ implies that $U_b''(\zeta) > 0$ and hence ζ is a strict local minimum for $U_b(.)$. Therefore, by parts (2) and (4) of Lemma (3), we can conclude for a large n, U_n also has a local minimum in the neighborhood $(\zeta - \delta, \zeta + \delta)$, which contradicts part (2) of Lemma 3. Consequently, $U_b'(y) > 0$ for all $y < b$. This together with the results of Lemma 3 implies that $0 < U_n(y) \leq U_b(y) < U_b(b) = e^b - q$. This completes the proof of part (ii).

The above proof of part (ii) also implies that $U_b(y)$ is a non-negative and strictly increasing and hence the $\lim_{y\to-\infty} U_b(y)$ exists and finite. We intend to show that this limit is zero. For this, we consider the differential equation:

$$\frac{\sigma^2}{2}U_b''(y) + (\tilde{r} - \frac{\sigma^2}{2} + \phi(y))U_b'(y) - (\tilde{r} + \epsilon + \phi(y))U_b(y) = 0 \tag{2.62}$$

Using the facts that $U_b'(y) > 0$, $\phi(y) > 0$, $\epsilon > 0$ and $(\tilde{r} + \epsilon + \phi(y)) > 0$ for $y < b$, we have $\frac{(\tilde{r} - \frac{\sigma^2}{2} + \phi(y))}{(\tilde{r} + \epsilon + \phi(y))} < 1$. We can use this inequality to deduce the following:

$$\frac{\sigma^2}{2}\frac{U_b''(y)}{(\tilde{r} + \epsilon + \phi(y))} + U_b'(y) \geq U_b(y)$$

Next, we keep y fixed and choose a $c < y$. Then, by integrating, we get:

$$\frac{\sigma^2}{2}\int_c^y \frac{U_b''(u)}{(\tilde{r} + \epsilon + \phi(u))}du + U_b(y) - U_b(c) \geq \int_c^y U_b(u)du \tag{2.63}$$

Using integration by parts and $U_b'(y) > 0$, $\phi'(y) < 0$, we derive,

$$\int_c^y \frac{U_b''(u)}{(\tilde{r} + \epsilon + \phi(u))}du = \frac{U_b'(y)}{(\tilde{r} + \epsilon + \phi(y))} - \frac{U_b'(c)}{(\tilde{r} + \epsilon + \phi(c))} \tag{2.64}$$

$$+ \int_c^y \frac{U_b'(u)\phi'(u)}{(\tilde{r} + \epsilon + \phi(u))^2}du \leq \frac{U_b'(y)}{(\tilde{r} + \epsilon + \phi(y))} \tag{2.65}$$

We can combine the above derived inequalities (2.63) and (2.65) to deduce the following bound:

$$\frac{U_b'(y)}{(\tilde{r} + \epsilon + \phi(y))} + U_b(y) \geq \int_{-\infty}^y U_b(u)du \tag{2.66}$$

Since $U_b(.) > 0$ on $(-\infty, y)$, we can conclude that the integral $\int_{-\infty}^y U_b(u)du$ is convergent. Moreover, $U_b(.)$ is strictly increasing and positive, hence we can conclude that $\lim_{y\to-\infty} U_b(y) = 0$. If there exists another bounded solution $\hat{U}_b(y)$ to the boundary value problem on the interval

$(-\infty, b]$, using Ito's lemma we can obtain the stochastic representation (2.47) of the solution and hence $\hat{U}_b(y)$ coincides with $U_b(y)$ on the interval $(-\infty, b]$. Therefore, the uniqueness follows and that completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

At this point, it is instructive to recapitulate what have been proven so far. With that detail laid out, it would be easier to keep track of the direction we are proceeding, having the goal in mind. It has been shown that, within the interval $(-\infty, b]$, the boundary value problem specified in the equation (2.46) has a unique solution $U_b(y)$. Moreover, the unique solution has the stochastic representation (2.47) and the following properties:

$U_b(y) > 0$ and $U_b'(y) > 0$ for all $y < b$. Moreover, $U_b(y)$ is bounded on $(-\infty, b]$.

Next, we extend each $U_b(.)$ to the interval $(-\infty, \infty)$ in such a way that it satisfies the linear differential equation $\mathcal{H}U_b(y) = 0$, and examine the properties of such an extended function to address the issue of existence. The next result documents that, such an extended $U_b(y)$ continue to exhibit the properties $U_b(y) > 0$ and $U_b'(y) > 0$ and $U_b(y)$ has no local maxima or minima on the entire interval $(-\infty, \infty)$.

Proposition 7. *Consider any b such that $\hat{y} > b > \ln(q)$, and the function $U_b(.)$ is extended to $(-\infty, \infty)$ so that it satisfies the linear differential equation $\mathcal{H}U_b(y) = 0$. Then,*

$$U_b'(y) - U_b(y) \geq \frac{2\epsilon}{\sigma^2} \int\limits_{-\infty}^{y} e^{-\frac{2}{\sigma^2} \int\limits_{u}^{y} (\tilde{r} + \phi(s))ds} U_b(u)du \qquad (2.67)$$

holds for every $y \in (-\infty, \infty)$. Consequently, $U_b'(y) > 0$ and $U_b(y)$ has no local maxima or minima on the entire interval $(-\infty, \infty)$.

Proof. Extending the function $U_b(y)$ to $(-\infty, \infty)$ in such a way that it satisfies $\mathcal{H}U_b(y) = 0$, we have,

$$\frac{\sigma^2}{2}U_b''(y) + (\tilde{r} - \frac{\sigma^2}{2} + \phi(y))U_b'(y) - (\tilde{r} + \epsilon + \phi(y))U_b(y) = 0 \qquad (2.68)$$

To reduce this second order linear homogeneous equation into a first order one, let $H_b(y) = $

$U_b'(y) - U_b(y)$. Using this transformation, the differential equation reduces to,

$$\frac{\sigma^2}{2}H_b'(y) + (\tilde{r} + \phi(y))H_b(y) = \epsilon U_b(y)$$

Multiplying this equation by the integration factor $e^{\frac{2}{\sigma^2}\int_c^y (\tilde{r}+\phi(s))ds}$ and integrating it, we obtain,

$$H_b(y) = H_b(c)e^{-\frac{2}{\sigma^2}\int_c^y (\tilde{r}+\phi(s))ds} + \frac{2\epsilon}{\sigma^2}\int_c^y e^{-\frac{2}{\sigma^2}\int_u^y (\tilde{r}+\phi(s))ds}U_b(u)du \qquad (2.69)$$

where, $c < y$ is any real number. We can choose c in such a way that $U_b'(c) > 0$. It immediately follows that $H_b(c) = U_b'(c) - U_b(c) > -U_b(c)$ and therefore,

$$H_b(c)e^{-\frac{2}{\sigma^2}\int_c^y (\tilde{r}+\phi(s))ds} \geq -U_b(c)e^{-\frac{2}{\sigma^2}\int_c^y (\tilde{r}+\phi(s))ds}$$

Letting c tend to $-\infty$, we have the following inequality.

$$\lim_{c\to-\infty} H_b(c)e^{-\frac{2}{\sigma^2}\int_c^y (\tilde{r}+\phi(s))ds} \geq \lim_{c\to-\infty} -U_b(c)e^{-\frac{2}{\sigma^2}\int_c^y (\tilde{r}+\phi(s))ds} = 0 \qquad (2.70)$$

Thus, using (2.70) in (2.69), we have the conclusion that proves the first part of the proposition.

$$H_b(y) = U_b'(y) - U_b(y) \geq \frac{2\epsilon}{\sigma^2}\int_{-\infty}^y e^{-\frac{2}{\sigma^2}\int_u^y (\tilde{r}+\phi(s))ds}U_b(u)du$$

The remaining part of the proposition is a straightforward implication of this inequality we just derived, (2.67).

To illustrate, let us assume there exists a point η such that $U_b'(\eta) = 0$. This implies that:

$$H_b(\eta) = -U_b(\eta) \geq \frac{2\epsilon}{\sigma^2}\int_{-\infty}^{\eta} e^{-\frac{2}{\sigma^2}\int_u^y (\tilde{r}+\phi(s))ds}U_b(u)du \qquad (2.71)$$

This above mentioned equation leads us to,

$$U_b(\eta) \leq -\frac{2\epsilon}{\sigma^2} \int\limits_{-\infty}^{\eta} e^{-\frac{2}{\sigma^2} \int\limits_{u}^{y}(\tilde{r}+\phi(s))ds} U_b(u)du \tag{2.72}$$

which is clearly a contradiction, since $U_b(y) > 0$ for all $y \in (-\infty, \infty)$. Therefore, we conclude that $U_b'(y) > 0$ for all y and $U_b(y)$ has no local maxima or minima on the entire interval $(-\infty, \infty)$. $\qquad\square$

So far, we have shown that the function $U_b(y)$ is positive, increasing and has no local maxima or minima on the entire interval $(-\infty, \infty)$. The function, by construction, has an intersection point with the curve $(e^y - q)$ at $y = b$. To address the issue of existence of exercise boundary successfully, it is enough to show that for a large y, we have $U_b(y) > (e^y - q)$. This will enable us to show that the function $U_b(y)$ that satisfies $\mathcal{H}U_b(y) = 0$ intersects the stopping boundary $(e^y - q)^+$ twice. Consequently, we consider constant multiples of such a solution $U_b(y)$ to obtain a b^* so that the corresponding $U_{b^*}(.)$ meets the stopping boundary tangentially at a single point, which is the desired *optimal exercise boundary*.

To show the graph of the function $U_b(y)$ intersects that of $(e^y - q)^+$ at least twice in the interval $[\ln q, \infty)$, it is sufficient to show that there exists a large y that satisfies $e^{-y}U_b(y) > 1 - qe^{-y}$. The next proposition addresses this remaining bit of existence issue under certain parametric restrictions.

Proposition 8. *If $\frac{-2\tilde{r}}{\sigma^2} > 1$, then $\lim_{y\to\infty} e^{-y}U_b(y) = +\infty$.*

Proof. From the previous lemma and equation (2.67), we know the function $U_b(y)$ satisfies the following property:

$$U_b'(y) - U_b(y) \geq \frac{2\epsilon}{\sigma^2} \int\limits_{-\infty}^{y} e^{-\frac{2}{\sigma^2} \int\limits_{u}^{y}(\tilde{r}+\phi(s))ds} U_b(u)du$$

Multiplying both sides of the inequality by the integrating factor e^{-y} and integrating it, we obtain,

$$e^{-y}(U_b'(y) - U_b(y)) \geq \frac{2\epsilon}{\sigma^2} e^{-y} \int\limits_{-\infty}^{y} e^{-\frac{2}{\sigma^2} \int\limits_{u}^{y}(\tilde{r}+\phi(s))ds} U_b(u)du$$

or,

$$\frac{d}{dy}(e^{-y}U_b(y)) \geq \frac{2\epsilon}{\sigma^2}e^{-y}\int\limits_{-\infty}^{y} e^{-\frac{2}{\sigma^2}\int\limits_{u}^{y}(\tilde{r}+\phi(s))ds}U_b(u)du$$

or,

$$e^{-y}U_b(y) \geq e^{-b}U_b(b) + \frac{2\epsilon}{\sigma^2}\int\limits_{b}^{y} e^{-u}(\int\limits_{-\infty}^{u} e^{-\frac{2}{\sigma^2}\int\limits_{x}^{y}(\tilde{r}+\phi(s))ds}U_b(x)dx)du$$

Since we consider y large and $\lim_{y\to\infty}\phi(y) = 0$, we can ignore $\phi(y)$. Also, $e^b U_b(b) = 1 - qe^{-b}$. Additionally, since $U_b(y)$ is positive and increasing, the value of this function is bounded below by $U_b(b)$. Thus, incorporating all these boundary values, we obtain:

$$e^{-y}U_b(y) \geq 1 - qe^{-b} + \frac{2\epsilon}{\sigma^2}\int\limits_{b}^{y} e^{-u}(\int\limits_{-\infty}^{u} e^{-\frac{2}{\sigma^2}\int\limits_{x}^{y}(\tilde{r}+\phi(s))ds}U_b(x)dx)du \tag{2.73}$$

Finally, we note that, if $\frac{-2\tilde{r}}{\sigma^2} > 1$,

$$\lim_{y\to\infty}\frac{2\epsilon}{\sigma^2}\int\limits_{b}^{y} e^{-u}(\int\limits_{-\infty}^{u} e^{-\frac{2}{\sigma^2}\int\limits_{x}^{y}(\tilde{r}+\phi(s))ds}U_b(x)dx)du = +\infty \tag{2.74}$$

Thus, we have the result of the proposition.

$$\lim_{y\to\infty} e^{-y}U_b(y) = +\infty$$

This completes the proof of the proposition. □

The above proposition shows that if $\frac{-2\tilde{r}}{\sigma^2} > 1$, $\lim_{y\to\infty} e^{-y}U_b(y) = +\infty$. Therefore, for a given q, we can choose a $b_0 > \ln q$, such that $U_{b_0}(b_0) > e^{b_0} - q$. Consequently, there is a δ such that $e^y - q > U_{b_0}(y)$ for all $y \in (b_0 - \delta, b_0)$. Also, $U_{b_0}(y) > 0 > e^y - q$ when $y < \ln q$. The above logical reasoning confirms the fact that the graph of the function $U_{b_0}(y)$ intersects the curve $(e^y - q)$ at least twice in the interval $[\ln q, b_0]$. A figure will be illuminating.

Figure 2.2   Family of Solutions
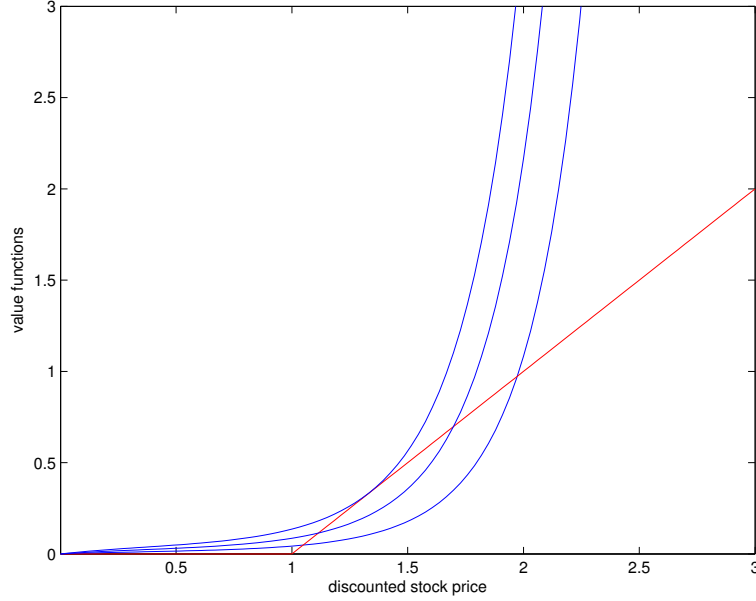
As shown in the figure, once we prove the intersection of these two curves, it is a matter of multiplying the solution of the differential equation with a constant until the shifted solution is tangent to the stopping boundary. The point of tangency is termed as *optimal exercise boundary* and represents the transformed value of discounted stock price where it is optimal for the owner to regain the stock. To accomplish this, we let,

$$k = \inf\{y : y > \ln q \text{ and } e^y - q = U_{b_0}(y)\} \tag{2.75}$$

Then, $\ln q < k < b_0$. Since $U_{b_0}(y) > 0 > (e^y - q)$ when $y < \ln q$, $y = k$ becomes the first point of intersection between these two curves. Therefore, $U'_{b_0}(k) < e^k$.

Since the differential equation is homogeneous, a constant multiple of the function gives the desired solution. This result is documented in the next theorem that completes the existence of the optimal exercise boundary.

Theorem 9.   (i) There exists a point $b^* > \ln q$ and a corresponding positive function $U_{b^*}(.)$ defined on $(-\infty, \infty)$ such that $\mathcal{H}U_{b^*}(y) = 0$ for all y, $U_{b^*}(b^*) = (e^{b^*} - q)$ and and $U'_{b^*}(b^*) = e^{b^*}$. Moreover, $U_{b^*}(y) = e^y - q$ for all $y \geq b^*$.

(ii) There exists a point x^* and a positive function $\hat{Q}$ defined on $(0, \infty)$, which satisfy all the conditions described in the Theorem 4.

Proof. Consider the point k and the function $U_{b_0}(.)$ described above. We consider the family of constant multiples of $U_{b_0}(.)$ given by $\{tU_{b_0}(.) : t \geq 1\}$, where t is a parameter. Since there exists a δ such that $e^y - q > U_{b_0}(y)$ for all $y \in (b_0 - \delta, b_0)$, and $U_{b_0}(y) > 0 > e^y - q$ when $y < \ln q$, clearly, there exists a $\delta_2 > 0$ such that the graph of $tU_{b_0}(y)$ also intersects that of $(e^y - q)$ at least twice in the interval (k, b_0), when $1 < t < 1 + \delta_2$. On the other hand, since $U_{b_0}(y)$ is strictly increasing, if we have $t > t_1 = \frac{e^{b_0} - q}{U_{b_0}(k)}$, then $tU_{b_0}(y) > (e^y - q)$ for all y in the interval $[k, b_0]$. Therefore, we introduce,

$$t^* = \inf\{t \geq 1 : tU_{b_0}(y) > (e^y - q) \text{ for all } y \in (k, b)\} \tag{2.76}$$

Then t^* is well defined and $1 + \delta_2 < t^* < t_1$. From the above analysis, it follows that the graph of $t^*U_{b_0}(y)$ intersects that of $(e^y - q)$ tangentially at some point on the interval $[c, b_0]$. Then we denote the point of tangency as b^* as follows:

$$b^* = \inf\{y : k < y < b_0 \text{ and } t^*U_{b_0}(y) = (e^y - q)\} \tag{2.77}$$

Then, it clearly implies that $t^*U_{b_0}(b^*) = (e^{b^*} - q)$ and $t^*U'_{b_0}(b^*) = e^{b^*}$. Since $\mathcal{H}U_{b^*}(y) = 0$ is a linear homogeneous differential equation and the function $t^*U_{b_0}(y)$ also satisfies all the conditions of the proposition 6 and therefore, coincides with the function $U_{b^*}(y)$. Therefore, all the conditions of the part (i) of the theorem are met by the point b^* and the function $U_{b^*}(y)$ and hence the result.

To prove part (ii) of the theorem, we simply use the reverse transformation $x = e^y$ and let $\hat{Q}(x) = U_{b^*}(\ln x)$ for all $x > 0$. Also, denoting $x^* = e^{b^*}$, we have x^* and the function $\hat{Q}$ satisfying all the conditions of the Theorem 4. Finally, we also have,

$$\lim_{x \to 0^+} \hat{Q}(x) = \lim_{y \to -\infty} U_{b^*}(y) = 0 \tag{2.78}$$

This completes the proof of the existence of the optimal exercise boundary. $\qquad \square$

To sum up, this section introduces the risk of bankruptcy according to the reduced form or, intensity modeling approach. We assume that default intensity is a function of discounted stock price and consequently, use a logarithmic transformation to facilitate our analysis. In this formulation, we prove the existence of an optimal exercise boundary. This optimal exercise boundary is of a threshold type and represents the transformed discounted stock price where it is optimal for the owner to exercise the option. Finally, the reverse transformation gives the optimal solution for the original problem and thus, we prove the existence of the analytical solution. In the next section, we use numerical methods to solve for the optimal stopping problem. Moreover, the sensitivity of the optimal exercise boundary to the relevant parameters of the system is analyzed, which is useful to address policy issues in the valuation of mortgage-backed securities.

### 2.4.7 Sensitivity Analysis using Numerical Methods

Theory of dynamic programming for optimal stopping problems suggests that the solution of the optimal stopping problem coincides with the smooth solution of the HJB equation, given by (2.37). This is the starting point of our numerical analysis. We use numerical methods to find a smooth solution of the HJB equation (2.37). To this end, we first numerically solve the linear homogeneous ordinary differential equation part of the HJB equation:

$$\frac{\sigma^2}{2}x^2 Q''(x) + (\tilde{r} + h(x))xQ'(x) - (\tilde{r} + \epsilon + h(x))Q(x) = 0$$

with the boundary conditions $\lim_{x \to 0} Q(x) = 0$ and $Q(b) = (b-q)^+$. This is essentially a boundary value problem. We use the Matlab boundary value problem solver ($bvp4c$ or $bvp5c$) to solve the problem. These solvers decompose the differential equation into a system of first order ODE's and use the collocation method to solve the differential equations and are quite robust in solving both linear and non-linear ODE's. Since the default intensity is a decreasing function of stock price, the differential equation is not defined at the left boundary point $x = 0$. However, since the left boundary condition is a limiting one, we take a value close to zero which successfully

serve the purpose.[9] For the right boundary, we consider a point that is higher than value of q. The initial choice of parameters are: $r = 0.02$, $\gamma = 0.07$, $\epsilon = 0.001$, $\sigma = 0.4$, $q = 1$ and $x \in [0.001, 3]$. With this parameter specification the following figure plots the *optimal exercise boundary*, denoted as $Xopt = 1.3169$.

Figure 2.3   Optimal Exercise Boundary
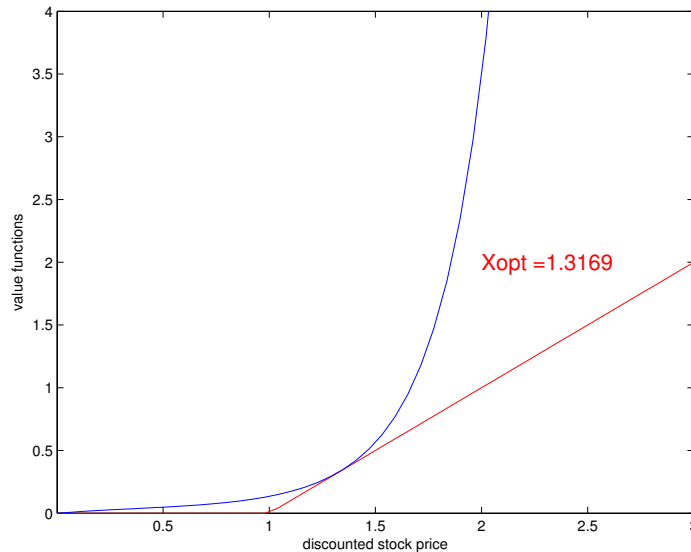
Next, we address the issue of sensitivity of the optimal exercise boundary. The comparative movement of the optimal stopping boundary in response to changes in the relevant parameters of the system is presented in Table 1. Each column of the table shows the values of the corresponding parameters and the very last column shows the resulting optimal boundary, $Xopt$. The table precisely documents the resulting effect of relative strength of the counteractive forces on optimal exercise boundary. An upshot of the findings and the relevant policy issues are addressed subsequently.

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[9]To address the issue in general, this condition gives rise to a singularity problem. Standard theory of convergence had been extended to BVP's to solve such problems (de Hoog and Weiss 1976, 1978). The idea behind it is to approximate the solution near the singular point by analytical means and use the approximate solution for the critical region (Shampine, Gladwell and Thompson, 2003).

Table 2.1   Sensitivity of the Optimal Exercise Boundary

Interest Rate (r)	Lending rate (γ)	$\tilde{r} = r - \gamma$	Volatility (σ)	Loan (q)	$X^*(\textbf{\textit{Xopt}})$
0.02	0.10	-0.08	0.40	1	1.3169
0.02	0.15	-0.13	0.40	1	1.3169
0.02	0.20	-0.18	0.40	1	1.3475
0.02	0.25	-0.23	0.40	1	1.3781
0.02	0.30	-0.28	0.40	1	1.3475
0.01	0.25	-0.24	0.40	1	1.3475
0.09	0.25	-0.16	0.40	1	1.3475
0.10	0.25	-0.15	0.40	1	1.3169
0.20	0.25	-0.05	0.40	1	1.3169
0.02	0.05	-0.03	0.44	1	1.3475
0.02	0.05	-0.03	0.48	1	1.3781
0.02	0.05	-0.03	0.52	1	1.4087
0.02	0.05	-0.03	0.56	1	1.4699
0.02	0.05	-0.03	0.60	1	1.5005
0.02	0.05	-0.03	0.4	0.5	0.8885
0.02	0.05	-0.03	0.4	1.5	1.7759
0.02	0.05	-0.03	0.4	2.0	2.2349
0.02	0.05	-0.03	0.4	2.5	2.7246

2.4.7.1   Discussion

First, we address the issue of the effect of change in interest rate (r) and lending rate (γ) on the optimal decision threshold. It is evident from the figures that minor changes in either of those rates does not have a significant effect on the boundary. The decision threshold starts responding only when these rates differ significantly from each other. To fix the idea, another column is added that documents the difference between interest rate and loan rate, which can be thought of as *effective rate of return* from this discounted stock price, $\tilde{S}_t$. We note that, the optimal decision threshold is crucially contingent on this *effective rate of return*, instead of the rates individually. When the lending interest rate is increased beyond $\gamma = 0.18$, keeping r fixed at 0.2, optimal exercise boundary tends to exhibit a steady upward movement. This trend continues until $\gamma = 0.25$ is reached. Beyond that point, optimal boundary is negatively related to γ. Similar effect can be generated by keeping γ fixed at a high level, and then gradually reducing interest rate r. Both this avenues lead us to the same conclusion that in this framework, *effective rate of return* $(\tilde{r})$ significantly affects the optimal exercise boundary. The intuition behind this effect can be traced back through the relative strength of the counter-

active forces that govern the dynamics of intensity modeling. This effective rate of return is a significant part of the expected rate of return from the stock as well as the discount factor. These two effect works in opposite direction in the decision to optimally exercise the financial derivative. Initially, increase in γ increases rate of return more than the discount factor and resulting effect on exercise boundary is positive. After the critical level of γ is reached, the dominating effect reverse in direction and consequently, we have the negative relation. The effect of change in volatility parameter, σ is comparatively much more intense. Increase in the value of the volatility causes a steady and steep rise in the optimal exercise boundary. This effect can be better explained using the transformed problem, described in equations (2.43) and (2.44). An increase in σ increases the volatility as well as causes a steep decline in expected rate of return. Since the return decreases faster, agent tend to wait longer in anticipation of capital gain from increased stock price. Finally, the optimal exercise boundary shows a dramatic increase in optimal threshold in response to an increase in loan amount (q). As the loan amount goes up, the exercise boundary shifts to the right, which means that discounted stock price has to be even higher for the rational agent to regain the stock. Now, intuitively, keeping all other policy variables of the economy unchanged, if the lender agrees to lend a higher amount for the same stock, that implicitly means higher loan amount maximize the expected return of the bank and the stock price has a probability to show upward trend. This effect coupled with rationality on the agents part tend to move the optimal exercise boundary upward.

2.5   Conclusion

In this chapter, we address the issue of valuation of a financial derivative when the underlying asset is subject to bankruptcy. The risk of bankruptcy is introduced according to both structural and reduced form modeling approach. In the structural form modeling, notion of bankruptcy is introduced following Black and Cox (1976) formulation where the asset is considered as bankrupt as soon as the asset price hits a lower boundary. Modeling the lower boundary as an increasing function of time, closed form expression for the valuation is derived in terms of *probability distribution of* the first passage time of Brownian motion and *the valuation of*

down and out barrier option. As a salient feature of this umbrella formulation, default time turns out to be a *predictable stopping time.* In the reduced form approach, default intensity is assumed to be a decreasing function of discounted stock price, which can be interpreted as conditional probability of default. The event of bankruptcy is modeled as a non-predictable phenomenon. The qualitative nature and the characteristics of the value function is rigorously studied and existence of an optimal exercise boundary is established. We also prove that this optimal exercise boundary is of threshold type and contingent on the policy variables that are treated as parameters of the system. We proceed further to use numerical methods to address the sensitivity analysis of the optimal exercise boundary. The results of our numerical simulation provide further insights into the linkage between optimal exercise boundary and the policy variables. We find that optimal exercise boundary is crucially contingent on the *effective rate of return* (defined as the difference between interest and lending rate) and exhibits a non-monotone relationship. However, the boundary is not so sensitive to the change in interest rate or lending rate until the difference is substantial. An increase in the Brownian volatility parameter causes a steep increase in the optimal decision threshold and magnitude of change is higher. We get an interval where optimal exercise boundary shows a monotone increasing relationship with an increase in volatility. The intuitions behind all the results are clearly explained. This numerical part of the research opens a scope for a potential future research that analyzes the dependence of the optimal decision boundary on the policy variables in a much more exhaustive manner. We also perform a sensitivity analysis with respect to the loan amount and get a positive monotone relationship. Therefore, to conclude, we note that introduction of credit risk results significant qualitative changes in the valuation of a financial instrument. Introducing intensity function as a decreasing function of discounted stock price is an alternative way that facilitated our analysis and produced insightful results. However, traditional beliefs about high interest rate or lending rate can be misleading in policy prescriptions in such a situation, primarily because of the dependence on the *effective rate of return.*

APPENDIX A.   Appendix to Chapter 1

Optimal resource allocation of agents R and P

The optimization problem of agent R in the second stage of first period is:

$$\max \ln C_{1R} + \beta \ln C_{2R} \tag{A.1}$$

$$\text{subject to } pY(1 - \tau) = C_{1R} + K_R + X_R \tag{A.2}$$

$$p'Y' = C_{2R} \tag{A.3}$$

$$p' = \frac{(X_R - G)^m}{(X_R - G)^m + (X_P - G)^m} \tag{A.4}$$

$$Y' = A_R K_R + A_P K_P \tag{A.5}$$

The interior optimality conditions are:

$$\frac{1}{C_{1R}} = \frac{\beta A_R}{Y'} \tag{A.6}$$

$$\frac{m(X_P - G)^m}{(X_R - G)\{(X_R - G)^m + (X_P - G)^m\}} = \frac{A_R}{Y'} \tag{A.7}$$

Analogus expressions for agent P are given by:

$$\frac{1}{C_{1P}} = \frac{\beta A_P}{Y'} \tag{A.8}$$

$$\frac{m(X_R - G)^m}{(X_P - G)\{(X_R - G)^m + (X_P - G)^m\}} = \frac{A_P}{Y'} \tag{A.9}$$

Denote, $\alpha = (X_R - G)^m + (X_P - G)^m$. Dividing (A.7) by (A.9), we get

$$\frac{m(X_P - G)^m}{(X_R - G)\alpha} \cdot \frac{(X_P - G)\alpha}{m(X_R - G)^m} = \frac{A_R}{A_P} \tag{A.10}$$

$$or, \left(\frac{X_P - G}{X_R - G}\right)^{m+1} = \frac{A_R}{A_P} \tag{A.11}$$

$$or, \frac{(X_P - G)}{(X_R - G)} = \left(\frac{A_R}{A_P}\right)^{\frac{1}{m+1}} . \tag{A.12}$$

Dividing (A.4) by $(X_R - G)^m$ we get

$$p' = \frac{1}{1 + \left(\frac{X_P - G}{X_R - G}\right)^m} \tag{A.13}$$

Substituiting the expression in (A.12) in (A.13) we get

$$p' = \frac{1}{1 + \left(\frac{A_R}{A_P}\right)^{\frac{m}{m+1}}} \tag{A.14}$$

Using the resource constraints and above formulation of p', we can reduce the FOC's of agents R and P as a system of linear equations in C_i and $X_i, i \in \{R, P\}$. The unique solution to the linear system is given by:

$$C_{1R} = \frac{1}{\beta(1 + m) + 2}\left[(p + (1 - p)\frac{A_P}{A_R})(1 - \tau)Y - (1 + \frac{A_P}{A_R})\tau Y\right] \tag{A.15}$$

$$C_{1P} = \frac{1}{\beta(1 + m) + 2}\left[(\frac{A_R}{A_P}p + (1 - p))(1 - \tau)Y - (1 + \frac{A_R}{A_P})\tau Y\right]$$

$$X_R = \left(\frac{1}{\left(\frac{A_P}{A_R}\right)^{\frac{m}{m+1}} + 1}\right)\frac{m\beta}{2 + \beta(1 + m)}\Delta + \tau Y$$

$$X_P = \left(\frac{1}{\left(\frac{A_P}{A_R}\right)^{\frac{m}{m+1}} + 1}\right)\frac{m\beta}{2 + \beta(1 + m)}\Delta\left(\frac{A_R}{A_P}\right)^{\frac{1}{m+1}} + \tau Y$$

where, $\Delta = [(p + (1 - p)\frac{A_P}{A_R})(1 - \tau)Y - (1 + \frac{A_P}{A_R})\tau Y]$. This concludes the derivation of the optimal consumption and resource allocation.

Proof of Proposition 2

We start by proving: If $\tau \leq \tau_{inv}$, positive investment equilibrium exists. We need to find a bound on τ such that $X_P - G \geq 0$, $X_R - G \geq 0$, $K_P \geq 0$, $K_R \geq 0$. From the expressions of appropriative investments from (A.15) we see that $X_R - G \geq 0$ if $\Delta \geq 0$. Now $\Delta \geq 0$ implies

$$\frac{1-\tau}{\tau} \geq \frac{1 + \frac{A_P}{A_R}}{p + (1-p)\frac{A_p}{A_R}} \; \forall \; \frac{A_P}{A_R} \in \left[\frac{A^L}{A^H}, \frac{A^H}{A^L}\right] \tag{A.16}$$

Taking limit on both sides as $\frac{A^H}{A^L} \to 1$ we have $\frac{1-\tau}{\tau} \geq 2$ this implies $1 - \tau \geq 2\tau$, or $3\tau \leq 1$, i.e. $\tau \leq \frac{1}{3}$. similar reasoning holds good for $X_P - G \geq 0$. Thus for $\tau \in [0, \overline{\tau}]$, where $\overline{\tau} = \frac{1}{3}$, equibrium effective appropriative investments are positive. We check the conditions under which $K_R, K_P \geq 0$. Substituting the values of X_R, C_{1R} in the expression of K_R we see that K_R reduces to

$$K_R = pY(1-\tau) - \left[\frac{m\beta}{2 + \beta(1+m)} \frac{\Delta}{\left(\frac{A_P}{A_R}\right)^{\frac{m}{m+1}} + 1} + \tau Y\right] - \frac{\Delta}{2 + (1+m)\beta} \tag{A.17}$$

Upon tedious manipulation we see that $K_R \geq 0$ implies

$$p(1-\tau) - a + b - \tau - \frac{(1-\tau)\Delta}{2 + \beta(1+m)} + \frac{(1+z)\tau}{2 + \beta(1+m)} \geq 0 \tag{A.18}$$

Where $a = \frac{m\beta(1-\tau)\Delta}{(2+\beta(1+m))(z)^{\frac{m}{m+1}}+1}$, $b = \frac{m\beta(1+z)\tau}{(2+\beta(1+m))(z)^{\frac{m}{m+1}}+1}$, $z = \frac{A_P}{A_R}$. Taking limit on both sides of the above equation as $\frac{A^H}{A^L} \to 1$ we have

$$\tau\left[-(1-p) + \frac{6 + 3m\beta}{2(2 + \beta(1+m))}\right] \geq \frac{m\beta + 2}{2(2 + \beta(1+m))} - p \tag{A.19}$$

If we assume, $\left[-(1-p) + \frac{6+3m\beta}{2(2+\beta(1+m))}\right] > 0$, we arrive at a condition that states $p < \frac{1}{2}$, which contradicts our basic assumption. Thus $-(1-p) + \frac{6+3m\beta}{2(2+\beta(1+m))} < 0$. By similar reasoning $\frac{m\beta+2}{2(2+\beta(1+m))} - p < 0$. Rearranging terms we see that $K_R \geq 0$ iff $\tau \leq \frac{\frac{m\beta+2}{2(2+\beta(1+m))} - p}{-(1-p) + \frac{6+3m\beta}{2(2+\beta(1+m))}}$. Let us call $\tau_1 = \frac{\frac{m\beta+2}{2(2+\beta(1+m))} - p}{-(1-p) + \frac{6+3m\beta}{2(2+\beta(1+m))}}$. Again, for $K_P \geq 0$, we substitute the values of X_P and C_P in the expression of K_P, which gives, $K_P = (1-p)(1-\tau)Y - c - \tau Y - d$ where $c =$

$\frac{m\beta\Delta[(1-\tau)}{2+\beta(1+m)}\frac{1}{(1+\left(\frac{A_P}{A_R}\right)^{\frac{m}{m+1}})}\left(\frac{A_R}{A_P}\right)^{\frac{1}{m+1}}$, $d = \frac{\Delta}{2+\beta(1+m)}$. Taking limit on both sides of the expression of K_P as $\frac{A^H}{A^L}$ goes to 1 we get $K_P \geq 0$ iff

$$(1-p)(1-\tau) - \frac{m\beta Y}{2+\beta(1+m)}[\frac{(1-3\tau)}{2} + (1-3\tau)] - \tau \geq 0$$

$$iff, \ \tau \leq \frac{2+m\beta+2(p-1)(2+\beta(1+m))}{-2(1-p)(2+\beta(1+m))+2+m\beta-2\beta}$$

Let $\tau_2 = \frac{2+m\beta+2(p-1)(2+\beta(1+m))}{-2(1-p)(2+\beta(1+m))+2+m\beta-2\beta}$. Thus $K_P \geq 0$ iff $\tau \leq \tau_2$. Thus for $\tau \leq \min\{\overline{\tau}, \tau_1, \tau_2\}$ all the three inequalities are satisfied. We denote $\tau_{inv} = \min\{\overline{\tau}, \tau_1, \tau_2\}$. Thus there exists positive levels of investment for $\tau < \tau_{inv}$ as, $\frac{A^H}{A^L} \longrightarrow 1$

Next, we prove the second part of the proposition. We show that an agent's best response to any choice of technology by the other agent involves in either choosing the best technology or the worst one i.e $A_i \in \{A^L, A^H\}$ for a given interval. Substituting the values of C_{1R} and C_{2R} into the utility function, we get $U_R = U_R(A_R, A_P)$. Differentiating U_R w.r.t A_R we get,

$$\frac{\partial U_R}{\partial A_R} \geq 0 \text{ iff } \frac{\tau}{1-\tau} \geq \Gamma(x) \tag{A.20}$$

provided $(1-\tau)(p+(1-p)x) - (1+x)\tau \geq 0$ and $\phi k(x)(1+x) + x - \beta \geq 0$. Here $\Gamma(x) = \frac{f(x)}{g(x)}$, $x = \frac{A_P}{A_R}$, $f(x) = (p+(1-p)x)[1+\phi k(x)] - (1+\beta)p$, $g(x) = \phi k(x)(1+x)+x-\beta$. Also, $\phi = \frac{m\beta}{m+1}$ $< \beta$, and $k(x) = \frac{1}{1+x^{\frac{m}{m+1}}}$. Now,

$$\Gamma'(x) \geq 0$$

$$iff, \ [\beta(2p-1) + \phi(1-\beta)(1-2p)k(x) + \phi^2(1-2p)k(x)^2 + \phi(1+\beta)(2p-1)k'(x)x \geq 0$$

$$iff, \ \beta x^{\frac{2m}{m+1}} + [(\beta-\phi)(1+\phi) + \beta(1-(\frac{m}{m+1})^2)]x^{\frac{m}{m+1}} + (\beta-\phi)(1+\phi) \geq 0. \tag{A.21}$$

The above is an equation of a parabola where both the roots, say x_1 and x_2, are negative. Therefore, for all $x \geq \max\{x_1, x_2\}$, the $\Gamma(x)$ is positively sloped. For the values of x that satisfy equations $(A.21)$, we get the best response of agent R is to choose either A^L or A^H. This interval of x implicitly put a restriction on τ. We denote that critical value of $\tau \geq \tau_R = \frac{1-p}{2-p}$.

Again, substituting C_{1P} and C_{2P} into the utility function, we get

$$U_P = U_P(A_R, A_P)$$

Following the same steps for the poor agent, we get if ,$(1 - \tau)(py + 1 - p) - \tau(1 + y) \leq 0$, and $(1 + \beta)(1 - p) - (py + 1 - p)(1 + \phi k(y) \leq 0$ then,

$$\frac{\partial U_P}{\partial A_P} \geq 0 \ iff \ \frac{1 - \tau}{\tau} \geq G(y) \ where, \tag{A.22}$$

$$G(y) = \frac{\beta - y - \phi k(y)(1 + y)}{(1 + \beta)(1 - p) - (py + 1 - p)(1 + \phi k(y))}, y = \frac{A_R}{A_p}$$

Now, $G'(y) \geq 0$

$$iff \ \ [\beta(2p - 1) + \phi(1 - \beta)(1 - 2p)k(y) + \phi^2(1 - 2p)k(y)^2 + \phi(1 + \beta)(2p - 1)k'(y)y \geq 0$$

$$iff \ \ (\beta - \phi)(1 + \phi)x^{\frac{2m}{m+1}} + [(\beta - \phi)(1 + \phi) + \beta(1 - (\frac{m}{m + 1})^2)]x^{\frac{m}{m+1}} + \beta \geq 0 \tag{A.23}$$

Which is again an equation of a parabola, where both the roots (say x_3, x_4) are negative, though different in values. Then, $x \geq \max\{x_3, x_4\}, G(y)$ is positively sloped. Thus, for $x \geq 0$,both $\Gamma(x)$ and $G(y)$ are postively sloped. Therefore, for the values of x that satisfy equation $(A.23)$, we get the best response for agent P is to choose either A^L or A^H.This interval of x implicitly put a restriction on τ. We denote that critical value of $\tau \leq \tau_P = \frac{p}{1-p}$.

Let $\frac{A^H}{A^L} \to 1$. If $\tau \in [\frac{1-p}{2-p}, \frac{p}{1+p}]$, then both the agents best response is to adopt either A^H or A^L. We denote $\tau_H = \frac{1-p}{2-p}$. From the Lemma 1, we know that positive investment equilibrium exists for $\tau \leq \tau_{inv}$. Thus for $\tau \in [\tau_H, \tau_{inv}], [A^H, A^H]$ can be sustained as a positive investment equilibrium. This completes the proof.

Proof of Corollary 1

Given (A^L, A^L) is an equilibrium in the bench-mark model the optimal choices of C_{1i} and $C_{2i}, i \in \{R, P\}$ are given as $C_{1i} = \frac{Y}{2+\beta(1+m)}, C_{2i} = \frac{\beta Y A^L}{2(2+\beta(1+m))}$. The SWF in this case is given

by

$$SWF(A^L, A^L) = (\ln C_{1P} + \beta \ln C_{2P}) + (\ln C_{1R} + \beta \ln C_{2R})$$

plugging in the values of C_{1i} and C_{2i} into the above equation we have

$$SWF(A^L, A^L) = \ln \frac{Y^2}{(2 + \beta(1 + m))^2} \left(\frac{\beta A^L Y}{2(2 + \beta(1 + m))} \right)^{2\beta}$$

Similarly when (A^H, A^H) is an equilibrium in the guard posting framework the optimal choices of C_{1i} and C_{2i}, $i \in \{R, P\}$ are $C_{1i} = \frac{Y(1-3\tau)}{2+\beta(1+m)}$ and $C_{2i} = \frac{\beta A^H (1-3\tau)Y}{2(2+\beta(1+m))}$ respectively. Plugging in the expressions of C_{1i} and C_{2i} in the social welfare function $SWF = U_R + U_P$ and rearranging the terms we get

$$SWF(A^H, A^H) = \ln \left[\left(\frac{(1 - 3\tau)Y}{(2 + \beta(1 + m))} \right)^2 \left(\frac{\beta A^H Y (1 - 3\tau)}{2(2 + \beta(1 + m))} \right)^{2\beta} \right]$$

From this it follows that

$$SWF(A^H, A^H) \geq SWF(A^L, A^L)$$

$$if \quad \left(\frac{(1 - 3\tau)Y}{(2 + \beta(1 + m))} \right)^2 \left(\frac{\beta A^H Y (1 - 3\tau)}{2(2 + \beta(1 + m))} \right)^{2\beta} \geq \frac{Y^2}{(2 + \beta(1 + m))^2} \left(\frac{\beta A^L Y}{2(2 + \beta(1 + m))} \right)^{2\beta}$$

$$\Leftrightarrow \quad (1 - 3\tau)^2 \left(A^H \right)^{2\beta} (1 - 3\tau)^{2\beta} \geq \left(A^L \right)^{2\beta} \Leftrightarrow \frac{A^H}{A^L} \geq \left(\frac{1}{(1 - 3\tau)^{2(\beta+1)}} \right)^{\frac{1}{2\beta}}$$

This completes the proof.

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